Language and communication in mathematics

Part 4
Analysing interaction processes with jigsaw during mathematics lessons in elementary school

The interaction processes that stem from the Jigsaw cooperation form can be analysed by decomposing the recipients’ roles based on their interactional status and their interpersonal speech acts. In this paper we elaborate on the influence of an individual's participation form for the ongoing interaction process.

Introduction
In the mathematics education discussion during recent years, it has repeatedly been pointed out that the learning of mathematics occurs in the interaction between independent, actively discovering and cooperatively communicative processes. It is thereby emphasised that learning on one’s own should be combined with learning from and with each other. In order to implement these requirements into everyday classroom, structured cooperative forms of teaching and learning are therefore increasingly realized when teaching mathematics in elementary school. In this article we deal with processes of interaction that stem from such a realization in the everyday classroom practice. Particularly, we deal with the use of the Jigsaw cooperation form in Geometry lessons in a second grade class (about 8 years of age). This cooperative form was conceived by Elliot Aronson in the early 1970s and has since then been extensively tested and researched (Aronson & Patnoe 1997; Kronenberger & Souvignier 2007). It is a form of mutual teaching and learning, in which the learners first develop a partial area of the topic that is being taught within groups of experts and then mutually communicate this expert knowledge that they have independently acquired in puzzle groups. The puzzle groups thereby consist of learners who have previously developed various partial areas. Our intention is to extend an existing theoretical and methodological framework (Brandt 2006, Krummheuer 2007) and to include some useful sociological insights coming mainly from role theory (Tatsis & Koleza 2006) in order to elaborate the influence of the individual participation form for the ongoing interaction process.

Theoretical framework
In the current discussion, the forms of group work in the pre-school and elementary school are mostly justified on educational goals and constructivist learning theories, whereas in terms of knowledge-oriented learning processes the small groups serve the learner as a place in which his/her own, subjective cognitions can be tested in terms of their viability. On this basis, the communication with others offers the potential for cognitive conflicts that can lead to a further development of the individual cognitions. In this sense, work within small groups especially
provides better learning conditions through improved possibilities for active, productive participation. Our perspective on the classroom processes is an interactionistic one (Bauersfeld 1994), in which the interaction serves as a place for joint construction of meaning. The individual cognition is thus bound to the participation in collective generative processes as interpretations that are seen as being shared. From this perspective Brandt and Krummheuer have developed a model of the everyday mathematics classroom situation (Brandt 1998, 2006; Krummheuer 2007): The way students are involved in explaining, reasoning and justifying content-related actions is crucial to the success of their learning – and this involvement includes productive and receptive aspects of the interaction. Referring to Goffman (1981) Brandt and Krummheuer identify these two aspects as the production-design and the reception-design of the processes of argumentation and describe tuition as a meshing of a smooth period of interaction, which is subject to the financial aspects of conflict minimization in communication, and condensed periods of interaction, which provide optimised enabling conditions for (content-oriented) learning. The condensed periods of learning differ from the smooth periods of interaction, on the one hand in terms of the complexity and explicitness of the argumentation that is put forth, and on the other hand in terms of the involvement of the listeners in this argumentation and the requirements for a change from a listening to a speaking form of participation in the process of negotiation. In collaborative learning situations the students adjust the balance between these two interaction modes by managing the turn-taking-problem and the content organisation of the interaction – and by doing this interactively, they shape their own opportunities for content-related learning processes (Brandt 2006).

Additional to these basic conversational tasks, the participants are simultaneously involved in profiling themselves and the others, which is described as face-work by Goffman. For example, in the classroom the teacher is caught between the need to correct a student and the need to respect the student’s face. The students are also caught between the need to participate actively in the classroom processes and the need to protect their face from any wrong or unpleasant moves. Face is defined as “the social value a person effectively claims for himself by the line others assume he has taken during a particular contact” (Goffman 1972, p. 5) and is considered one of the most important factors that affects a person’s behaviour, or, in sociological terms, a person’s role performance (Goffman 1971, p. 26). During the condensed interaction periods the students may interpret the ongoing situation differently; this leads to the adoption of different roles which, in turn, affect the argumentation processes in various ways. In the following section we will describe the methodology we have used in order to clarify the interaction processes involved in a Jigsaw cooperation situation.

Methodology

Classroom interactions have a fundamentally different structure of interaction than dyadically organised face-to-face dialogues, whether dealing with teacher-centred class discussions or partner and group discussions among students. All discussions in class are rather polyadically structured: the teacher-centred class discussion cannot be described as a dialog between a teacher and the learner as a homogenous counterpart, and in addition dialogs take place in public as determined by the classroom and have characteristics that are caused by the integration into the overall events in this public area (Brandt 1998).

In classrooms it is thus possible, as a speaker, to focus on individuals among a multitude of listeners by addressing them correspondingly. Goffman (1981) suggests the decomposition of the interaction into more detailed analytical elements than ‘speaker’ and ‘hearer’. Brandt and
Krummheuer have taken up this view and developed it further into a conceptual net for the description of educational interaction patterns. The starting point is the individual statements or actions that have a certain scope; the allocation of a certain listener role is thus not based on the quality of listening, but on the speaker’s way of addressing (Brandt 1998).

In multi-party interactions some of the persons are directly involved in the process of negotiation within the scope of the statement; within this direct participation, individual recipients can be emphasised by being named, through personal pronouns, or else through gestures or facial expressions. This exclusive role is called the interlocutor and is for example connected with a special right to speak in the following turn, but also with a certain obligation to react to the current statement. In smaller groups all recipients may be given the status of interlocutor. This must be differentiated from the status of listener, who has equal access to the reception of the negotiation, but is not obliged to provide a reaction that goes beyond providing the signals of a listener. In class discussions the differentiation between the interlocutor and the listener can be found for example in the relationship between the teacher and the student who has just been called upon, whereas the rest of the class is often conceived of as a group of listeners. It is, however, possible to fall back on these listeners if there is no reaction on the part of the interlocutor who has been addressed or if his/her reaction is deemed to be insufficient. In public places, in addition to the interlocutor and the listeners, there are usually other persons present within the framework of the scope of a statement, who are not directly involved in the process of negotiation. These recipients can be further differentiated into overhearers and eavesdroppers. Overhearing is definitely tolerated by those directly involved, but overhearers do not have the right to ask about details that make it possible to gain an understanding access to the process of negotiation. Overhearers are taken into consideration when it comes to the choice of words, the choice of the topics that are discussed or the degree of indexicality. The status of eavesdropper is indicated by the speaker through the posture, the pitch of voice and/or through an inaccessible language code; the understanding reception of the statement is thus made difficult or impossible. In this model, eavesdropping is an allocation that is made by the speaking person and his/her immediate interlocutors, for example when children in class switch from the teaching language (e.g. German in the observed classroom) to their language of origin and thus exclude individuals present from the understanding reception.

These categories are not always selective; there are, especially within each main category, grey zones of addressing that can be clarified only by the subsequent conversation. In processes of group work, interlocutor and listener can be assigned to the interior structure of a condensed period of interaction, whereas overhearer and eavesdropper represent different outer relationships. If a change occurs from the status of an overhearer to that of a listener or even interlocutor, a change occurs in the group affiliation. From the role theory point of view, these shifts between categories do not happen casually; by accepting the fact that students’ behaviour may be attributed to ‘face-saving’ strategies (Brown and Levinson 1987) we are then able to observe the interpersonal and intrapersonal function of each utterance. By doing this, we are then able to draw some conclusions on the group dynamics, its evolution in time and its relation to other groups’ dynamics. In the following section, we deal with the inner structure and dynamics of a single study group.

Sample analysis

The teaching unit was integrated into the normal everyday teaching and was documented with two video cameras. The processes of cooperation by two groups of experts and one associated core group were video filmed and from this recording, sections were transcribed for analysis. Before that, the class had gained experience realising the group jigsaw form of coope-
ration in social studies, whereas more complex cooperative assignments in mathematics education were not yet dealt with. Within the groups of experts and without a preparatory phase, the learners were supposed to find possibilities to jointly or individually record on paper three-dimensional structures made of wooden cubes. As an introductory problem, the teacher had specified a concrete cube structure (see Figure 1) and connected it with a little story. There were various hints for the groups of experts in order to lead to the following solutions: drawings of the various side views, verbal formulation of a construction manual in a construction dictate and notation in building plans, whereby the heights of the structures were recorded in a square matrix (see Figure 1).

![Figure 1. Cube structure and expected solution in the building plan](image)

The expert group in focus (Charline, Jamal, Jens and Nele) were to deal with the third solution possibility, i.e. with the notation in the building plan. For this, the children had a worksheet with a 3x3 matrix as a guideline, together with the notice: “This space is sufficient” and the tip: “Place the cubes on the plan and circumscribe them with a pencil”. Due to space limitations we present only a part of the discussion, while our analysis deals with the whole episode (1–55). The cells are named after the letters a, b, c, d, e, f, g, h, i.

9. N: and look/ then we would draw around it here/ [moves with the pen around the construction; only indicates the drawing] then we would make lines in the middle like this/ [indicates lines along the inner edges] well and how we would have to build it higher/ [voice becomes quieter] I wouldn’t know either/
10. J: [raises his right hand] I know it-
11. C: <I got it
12. N: <how/ [looks at Jamal]
13. J: you you put a few more inside, here, these two [momentarily lifts the two top cubes off of cell a]
14. C: look/ I had the idea
15. C: <look/ [Nele looks towards Charline] here we can/ then this is the top one/ [leans onto the table, shows the top cube cell a] then the second one goes [shows the middle cube cell a] there and then we can draw that/ [removes all cubes from the diagram]
16. J: <so like this/ [draws three concentric squares onto the cover sheet of the group assignment]
17. J: [looks up from his drawing back to Charline and Nele]
18. C: so like this/ these are/ (...) so that they’re [places the two cubes back onto the diagram (1,0,0),(1,0,0),(0,0,0)] top of one another somehow/ [places the rest of the cubes onto the diagram] and then we can like draw that there too-
19. N: <yeah- but we don’t have enough room for that/
20. C: <|arranges the cubes on the diagram (1,1,1),(1,1,1),(0,0,0)]
21. J: yeah we can do it [points to his drawing] like this-
22. Je: [has been building with his cubes at his seat until now] or we draw that would work too/ if we/
[23 Jens is still presenting his solution; Nele is listening; Charline asks Jamal to erase his
drawing; Jamal erases his drawing]
26. Je: #the stones/ for example give them different numbers and then uhm put them on top of each
other\ (\ ) [becoming quieter] and then write down the numbers in the order#
28. C: #exactly we <can-
29. N: <I didn’t really get that
30. C: well look/ we could give numbers to the cubes/ [takes the cubes off the diagram]#
31. N: #oh like like for example [counting the cubes while speaking] this is one that is two that is
three that is four that is five that is six#

As a starting point for the group work, Nele draws the depicted building on the empty
building plan and circumscribes it with a pencil; she thereby converts the hint for the solution
into a concrete action and finally formulates their task (9). She thus focuses on the problem of
recording the third dimension (“to build it higher”) on the two-dimensional form, which was also
not solved by the tip. All other children in the group are uniformly addressed by her as interlo-
cutors and could take the next turn. Nele’s initiating act places her in the lead of the discussion
and thus makes her a collaborative initiator (Tatsis & Koleza, 2006), i.e. a person that takes
initiatives but at the same time respects the other participants’ opinions. During the next two to
three minutes, three different possibilities arise for depicting the third dimension. The entire
scene can be described as a chain of condensed periods of interaction, which focuses on the
problem of three-dimensional constructions and thus answers the question that Nele directed to
everyone as interlocutors. First Jamal replies (10), and fulfils Nele’s request to first of all verbally
explain the suggestion. Later he draws concentric squares on the sheet in front of him, represen-
ting the tower and which can be compared to topographic lines (see Figure 2).

Charline, however, does not feel obliged to follow his suggestion attentively as an interlocu-
tor but is more intent on gaining their attention as interlocutors by repeatedly saying “look” (14,
15) and by construction activities on the building plan which is lying in front of Nele. The fact
that she is so eager to present her view that she ignores Jamal’s contribution reveals her inten-
tion to maintain her face in the group, by taking a more active stance in the discussion. Her
solution can be interpreted as an attempt to draw the floor plan and the upright projection of the
building. From a mathematical point of view, a further differentiation of this solution would be
quite correct, but it would be not compatible with the notice. This space is insufficient, and Nele
comments on the suggestion correspondingly (19). Nele is thus addressed both by Jamal and by
Charline as interlocutors regarding possible solutions to her initial question, whereas the other
children are only involved as listeners or they feel as if they are addressed as listeners and thus
do not feel obliged to comment. Jamal finally focuses his gaze on the negotiations between
Charline and Nele and presents his finished drawing to Nele at a suitable moment (21). However,
almost simultaneously, Jens uses the opportunity to offer his solution to her (22). Although he
makes it clear that he has heard the other suggestions, his contribution is not argumentatively
connected with the suggestions that were presented by Jamal and Charline. Instead he seems to bypass these two solution suggestions thematically, and to react directly as a speaker to Nele’s initial question; this corresponds to the role of the listener for the previous answers to the initial question. This variety of suggested solutions clearly shows that the interaction patterns in this group are structured in such a way that all participants have the opportunity to express their views; this ‘exchange’ between the various recipients’ roles helps to release the potential tension, since everybody has the right to speak and justify his/her position. Shortly thereafter, Charline is the first to react affirmatively to Jens’s idea that his interlocutor Nele has not understood yet (29). Following this unspecific question in a mutual, dyadically oriented discussion partnership, Charline and Nele clarify how Jens’s idea for a solution is to be understood (30–33); Jens is thereby simply included as a listener. Finally, Jens and Jamal take turns and clarify in which cell the numbers 5 and 6 should be entered (Figure 2). These four students can be characterised as collaborative evaluators (Tatsis and Koleza 2006) concerning Jens’s solution, since their mutual involvement is based on the evaluation and enrichment of each other’s suggestions. The solution is thereby accepted as a group solution and the boundary to the group of listeners consisting of the other children is again dissolved in favour of a general group of interlocutors within the group. After a further brief clarifying sequence, which is negotiated especially between Nele and Jamal (44–52), Jens finally announces: “we’ve we’ve figured it out” (55). With this speech act, people outside the group, and particularly the teachers, are for the first time addressed as interlocutors. The first internal solution process is thereby concluded in a way that is visible and audible to the outside.

Conclusions
The episode that we have analysed here can be described through the participation theory model of the smooth periods of interaction and the condensed periods of interaction as optimised learning situations – at least for individual participants – regardless of a measurable learning success. However, every participant brings very individual orientations into the cooperation, and the group work process is a space of cooperation, which is characterised by the interdependency of individual participation profiles (Brandt 2006). These participation profiles, seen from role theory, may be attributed to ‘face-saving’ strategies. Sometimes, these strategies hinder the cooperation (Tatsis & Koleza 2006); this was not the case in our focus group and this may be attributed to Nele’s moderating participation, being a collaborative initiator at the beginning and a collaborative evaluator later on, a fact that had a positive influence on the group’s cooperation. However, to what extent these individual orientations and the resulting dynamics that arise in group interaction are, if at all possible, methodologically controllable, is probably more than only a question of the work material, the assignment or a communicative or cooperative competence that can be trained through questioning behaviour.

References


Discussing on the fairness of probabilistic games: The creation of a discursive community with kindergarten children

In the present paper we provide an analysis of the verbal interactions that took place during the realisation of two activities in a kindergarten classroom. Focusing on discussions about the fairness of the games played we observed the strategies that children used to justify their opinion and monitored the development of their intuitions concerning the fairness of a game.

Introduction

The importance of probabilistic thinking is quite high since in everyday life we have to make predictions and then decide under uncertain circumstances while having to evaluate at the same time a large amount of information. This fact is acknowledged by all mathematics educators, researchers and policy makers and is expressed in various papers or other official reports. For example, the guidelines given by NCTM (2000) stress the importance of having all students develop an awareness of probability constructs and applications from an early age. Moreover, research has shown that students have difficulties with these concepts (e.g. Fischbein & Schnarch, 1997; Shaughnessy, 1992). Concerning young children’s probabilistic thinking there has been a considerable amount of research (e.g. English, 1993; Jones et al., 1997; Kafoussi, 2004; Piaget & Inhelder, 1975) but the results sometimes seem to be contradictory. For example, the work of Piaget and his colleagues and the work of Fischbein seem to have crucial differences with respect to their educational implications (Greer, 2001; Skoumpourdi, 2003). Langrall and Mooney (2005) in their literature review state that the results of relevant studies show that children eventually can move beyond subjective judgements and animistic behaviours, although it is not always the case. The research described in this paper deals with a number of activities conducted for a European project and its basic aim was to examine the children’s and teacher’s language during these activities and particularly:

a) The ways young children verbally express their thinking, as they try to perceive the notion of the fairness of a game.

b) The teacher’s verbal actions in establishing the notion of the fairness of a game.

In the next section we will describe our theoretical framework related to the functions of language in mathematics education which led us to the adoption of a particular methodology for the analysis of the activities.

Theoretical framework

The role of communication is emphasised by a number of contemporary educational researchers, who sometimes follow distinct approaches. Whether the focus is on the student, the
community or the interactions that the student participates in, there is a common agreement on
the central role of communication. Concerning mathematics, there are various communicative
means: ordinary language, mathematics verbal language, quasi-mathematical language, symbo-
lic language, visual representations and unspoken but shared assumptions (Pirie, 1998). All
these characteristics, some of which are unique in the mathematics classroom, eventually lead to
the establishment of a discursive community which has its own practices and is regulated by
a specific set of norms (Yackel and Cobb, 1996). Relevant research in the field has shown that the
“unspoken but shared assumptions” (norms) in mathematics classrooms may influence the
content and the structure of the interactions that occur (Tatsis and Koleza, 2008; Yackel and
Cobb, 1996). Particularly, they may influence the way a student perceives (and therefore at-
ttempts to solve) a problem (Boaler, 1999). Concerning the role of language in this process we
believe that it is used to establish mathematical meanings during processes of joint negotiation.
These processes can be observed and analysed through the interactions that take place in such
settings and by looking for regularities or any kind of pattern that can be connected to the
establishment of a discursive community.

Methodology
The approaches that endorse the joint formulation of mathematical meanings or even the
establishment of a discursive community focus their analytic lens on the communication that
takes place in educational settings. Thus, there is a tendency that the focus should be the verbal
(and sometimes also the non-verbal) exchanges of the participants. Sfard (2006) talks about
communicational moves and practical actions which continuously affect and shape the interac-
tion. The focus of our paper are these verbal communicational moves and we will use all other
non-verbal actions as supplementary to our analysis. Following Austin’s (1962) terminology, for
the rest of the paper we will call these verbal moves speech acts. The basic assumption under-
lying the speech act theory is that each verbal expression is used to perform an act, therefore it
is interpreted as such by the listener. In our case, after transcribing the discussions into written
text we looked for any patterns in the group’s discussion that may imply the establishment of
a common new probabilistic notion (particularly the notion of fairness) and at the same time
looked for patterns in the group’s discussion that may reveal the establishment of a shared
assumption. We decided not to use any predetermined categories for the participants’ speech
acts, because our focus was their evolution and their commonalities.

The experiment took place in a typical state kindergarten school of Rhodes in 2007. The
children that worked under the guidance of an experienced teacher were five years old. The
tasks were designed to examine children’s informal knowledge of probability (Skoumpourdi,
Kafoussi & Tatsis, in preparation).

Sample analysis
We will present the analysis of two tasks; Task 1 concerns the prediction of the most/least
likely event in a random experiment and Task 2 is related to probabilities comparison.

Task 1 – Turtle vs. snail
Nineteen children were separated in two groups (the teacher was member of one team, so that
both teams had an equal number of members) and played the following game: the first group had
to move a turtle to the end of a ten-stepped green path; the second group had to move a snail to
the end of a ten-stepped red path, parallel to the turtle’s one. The spinner used had its sectors
painted green and red, at the proportion of $\frac{3}{4}$ and $\frac{1}{4}$ respectively. According to the colour that
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came out, the respective group made or didn’t make one step with their pawn. The winner was the team that will arrive first at the end of the path. The green team won and then the following discussion took place between the teacher and the students. The letter T refers to the teacher and all the other letters refer to the young students (each letter is used once). The symbol ( ) is used when many children speak simultaneously and our notes are in brackets:

1. T: I would like you kids now to tell me how did you make it and finish first?
2. Y: Because it was larger.
3. T: What was larger?
4. Y: The green!
5. S: The green.
6. T: Ah, so what does this mean?
7. P: What does it mean?
8. K: That we won.
9. T: You did it. Because you say that it was larger… Which was larger in the disc?
10. (): The green. [About six children speak]
11. T: So, if we wanted to play fair and complete the game? If we wanted the snail and the turtle to have equal chances to walk, should the disc be like that?
12. (): No! [Almost by all children]
13. T: So Yiannis, you tell us…
14. Y: Because it was, it was little orange. The green was, the green was more.

Then the children worked in two groups in order to construct a disc that would make the game fair. We can observe many interesting things in the discussions above. The teacher’s speech acts are focused on a basic aim: to help the children understand the relation between the area of the sectors and the fairness of the game. In order to achieve this aim she moves gradually: firstly she tries to relate the area of the sectors to the game they already played (9) and then she tries to introduce the notion of fairness and its relation with the area of the sectors (11). The children seem to realise from the very beginning that the sectors’ areas influence the outcome of the game, but they are unable to answer the question what does it mean (6). Although the teacher is probably expecting to hear something about the connection between the sectors’ areas and the outcomes of each team’s draws, the children do not seem ready to verbally express this relation. Then the discussion moves to the relation of the spinner’s sectors with the fairness of the game (11); the children need some time to grasp the idea that equal sectors mean equal chances, but they do realise that the game they already played was not fair (12) and that it has to do with the sectors’ size (14). Later on, the children were asked to construct a ‘fair spinner’. Figure 1 below shows the evolution of a group’s construction; initially, a female student separated the disc in two sectors as shown. Then, after the teacher’s query (“Is the spinner fair now?”), another line was added and a student had the idea of creating equal numbers of sectors; the outcome is shown in the fourth image of the sequence.

Figure 1. Constructing a ‘fair’ spinner
Task 2 – Cherry trees

Fifteen children were separated in two groups (the teacher was member of one team, so that both teams had an equal number of members). Each group used a spinner with the sectors painted green and red as shown in Figure 2.

There was one picture with two cherry-trees and every tree had ten cherries on it. One child from each group turned the spinner and according to the colour that came out, the group picked one cherry from their cherry-tree, when the spinner stopped at the green colour and two cherries from their cherry-tree, when the spinner stopped at the red colour. The winner was the team that will firstly collect all cherries. The game finished before all children had the chance to play. At that moment, the following discussion took place:

1. T: Which team won?
2. R: We did.
3. T: You did.
4. A: But it’s not fair.
5. T: Angelica says it’s not fair. How could you win so fast?
6. A: Because the red is more times.
7. T: Nicolas, do you agree? Angelica said that in their spinner there is more red and the game is not fair. [Nicolas does not reply]
8. Y: Because it is one, two, three. [Yiannis shows the red sectors in the right spinner]
10. Y: Because it is three pieces. And the green is two.

15. T: So? Why it is not fair Yiannis?
16. Y: Because it must have three, three, three green and three red.

22. T: Why this team had the red more times?
23. N: Because it’s bigger.
24. T: What is bigger? Please explain to us. [Nicolas gets up, goes close to the spinners and starts counting the red sectors of the right spinner]
25. N: One, two, three, one, two, three makes us six and these make four.
26. T: And what did we have on this side? [Nicolas moves towards the left spinner] Let Yiannis tell us. Here? What did we have Yiannis?
27. Y: Three green and two red.

41. T: How should these discs be like? Can you tell us Angelica?
42. A: It should be red-green, red-green…
43. T: So, both discs should be what? [Yiannis raises his hand] Tell us Yiannis. [Yiannis moves towards the discs]
44. A: The same.
The discussion went on with the teacher’s query “If you were about to play the game again, which spinner would you choose?” Some children initially chose the same disc that they already used, but gradually they agreed that they should pick the spinner that has ‘more red’. The previous discussion shows that some children have grasped the notion of the fairness of the game; in (4) Angelica says without being asked that the game was not fair. This utterance reveals an attitude established to the particular student that playing probabilistic games involves a discussion about their fairness. But we cannot conclude that this has become a shared assumption across the children. What looks like a shared assumption is that the fairness of a game is associated with counting and comparing the spinners’ sectors; without being explicitly asked to, two children in our case (Yiannis and Nicolas) compare the number of sectors by counting them. Yiannis focuses on half the spinner and sees three green and two red sectors in the left spinner (8, 10), while Nicolas is able to see the spinner as a whole, so he finds the sum of all sectors (25). Angelica, on the other hand, suggests a different approach, probably influenced by their work in the previous task (Figure 1): she suggests that the sectors should be green and red alternately, without any reference on numbers. The teacher chooses to focus not on the different alternatives presented, but on the fact that both discs ‘should be the same’. This is probably related to an assumption that there should be a ‘general agreement’ at the end of the school day on what has been learned.

Conclusions

The basic characteristics of the teacher’s verbal acts may be categorised according to their content and their intentions. Concerning mathematical content, in our case the teacher made speech acts that dealt with the notions of chance, fairness, straight lines (when the children were constructing a ‘fair’ spinner), equality and inequality, sectors, disc, circle, numbers (steps to be made) and space orientation (direction of the pawns). In most cases, she used quasi-mathematical language as well as children’s everyday language, which proved to be very efficient, since we did not observe any signs of misunderstanding. The relatively ‘new’ for the children notion of fairness was introduced in a functional – and not static – manner: a fair game is a counter-example of the unfair games already played and a fair game is made possible through the use of appropriate materials (spinners).

Concerning their intentions, the verbal acts were made in order to ensure that:

a) most (if not all) children will have the chance to talk and express their opinion
b) most (if not all) children will comprehend the concepts involved in this game
c) the ‘correct’ view will be accepted at least by the majority of children
d) the practical tasks involved (e.g. construction of a ‘fair’ spinner) will be completed successfully and on time

The above verbal acts did not always achieve their end; there were times – especially in the ‘Cherry-trees’ game – when many children seemed unable to follow the teacher’s guidance and see the difference between the two spinners and the respective outcomes. This in turn affected the establishment of a discursive community in which every participant can speak and has something to contribute on the issue under discussion. Moreover, the teacher sometimes focused too much on the ‘correct’ opinions, leaving queries and different opinions unanswered. As we already mentioned, this may be attributed to her assumptions on what ‘needs’ to be learned from these activities.

Most children had the opportunity to express their ideas and we could observe that counting proved to be their most powerful strategy for justifying their opinion. This was more obvious in
the ‘Cherry-trees’ game, where the children used different strategies – based on counting – in order to justify their answer about the fairness of the game. In our opinion, the occurrence of these verbal arguments reveals the ability of kindergarten children to engage meaningfully in probabilistic activities concerning the notion of the fairness of a game.

Langrall and Mooney (2005) suggest that instruction on probability should be two-fold in order to elicit the learner’s “awareness of the potential conflict between a primary intuition and the logical structures of probability (p. 113) and to assist the development of “more normative secondary intuitions that can be accessed to override inappropriate or limiting primary intuitions” (p. 113). We believe that in our case, the games the children played together with the teacher’s scaffolding assisted them on the development of some secondary intuitions concerning the fairness of game and the role that materials play on that. Moreover, we have witnessed the primary steps of a discursive community based on the premise that each opinion should be justified in order to be accepted by the more experienced members of the community, i.e. the teachers.

References

Change: What change? Considering whether a national initiative has had an impact on the quality of classroom talk in primary mathematics

This paper will consider the transcripts of two mathematics lessons in a primary school as 'telling cases' (Mitchell 1984) which illustrate the nature of communication in primary mathematics classrooms in the United Kingdom at present. Using qualitative analysis and a theoretical framework developed as part of an earlier study, the quality of the talk is examined and the nature of the social and mathematical dimensions of the talk analysed. Quantitative analysis explores the proportions of the talk that are attributed to the pupils and to their teachers and comparisons are made with earlier findings. The examples are selected to throw light on the quality of mathematical communication and the nature of participation in the classroom talk by the teachers and pupils involved.

Introduction

This paper arose as a result of observations made of primary mathematics classrooms during this academic year after a gap in making such observations of a period of over ten years. During this period in English schools, the National Numeracy Strategy was introduced and implemented. The National Numeracy Strategy came with an attendant programme of professional development for teachers delivered by Primary Numeracy Consultants and various changes in the structure and content of mathematics lessons were recommended. Resources were also made available to the teachers to support their planning including a comprehensive collection of ready made lesson plans following the recommendations of the Strategy. However the immediate impression that we gained of the lessons observed after this gap was that not a lot had changed in terms the quality of talk or its structure and so we decided to examine this impression in greater detail.

Background

The original study, with data collected between 1996 and 1998, was part of Jenni’s doctoral research project which sought to examine the quality of communication in primary mathematics classrooms. The data were gained through participant observation in such classrooms over extended periods of time and were analysed using grounded theory techniques and frameworks derived from discourse analysis. One of the key findings of the research was of a connection between the social component of the classroom talk and the mathematical challenge of the activities being offered to the children. Evidence of the children’s engagement in mathematical thinking and reasoning characterised by actions such as offering justifications, generalising and proving occurred in classrooms in which the children were able to voice their own ideas in an open social context and in which the activities offered to them were highly mathematically challenging.
The transcripts that we consider here were chosen from a selection of classroom observations collected in the autumn term of 2007. They came from classrooms in which the teachers were participating in a research project undertaken with support from the National Centre for Excellence in Teaching Mathematics and a north London Local Authority. The project involved the teachers in a continuing professional development course which comprised six half day workshops and follow up activities with the children in their schools. The transcripts and observation data formed part of the preliminary data gathering before the commencement of the course. So the agenda for collecting the second batch of data was unrelated to the first and the similarities which we observed came as a surprise to us. It is these surprising similarities that we felt needed examining and which have led to the analysis presented in this paper.

Theoretical framework

In Jenni’s original doctoral study she focused on the talk of the primary mathematics classroom and paid considerable attention to social and mathematical aspects of this talk. She based her analysis on ideas about language games and forms of life outlined by Wittgenstein (Wittgenstein, 1968). In essence she is interpreting language games as the use of language together with the actions that are woven into it. She also takes a form of life to be an established human social practice, established within a community and involving its own purposes, rules and behaviours as well as its own special language games. Mathematical forms of life are characterised by thinking and reasoning that emphasise exemplifying, specialising, changing, varying, altering, completing, deleting, correcting, generalising, conjecturing, comparing, sorting, organising, explaining, justifying, verifying, convincing and refuting about number, data, shape and space. They also involve making connections between mathematical ideas and concepts in a variety of contexts as part of the process of generalising mathematically. This process of generalising comprises conscious mathematical thinking and reasoning and the development of mathematical argument and includes notions of proof.

In her study Jenni focused on the talk of the primary mathematics classroom and paid considerable attention to social and mathematical aspects of this talk. She considers the social and the mathematical to be key to the development of children’s ability to participate in mathematical forms of life.

Jenni’s model suggests that every utterance in the talk can be analysed in relation to its social and mathematical components and we want to suggest that these components can be viewed as dimensions of the talk as illustrated by the following diagram:

useful sociological insights coming mainly from role theory (Tatsis & Koleza 2006) in order to elaborate the influence of the individual participation form for the ongoing interaction process.
The social dimension of talk is connected with building and maintaining the social relationships within the class, between teacher and pupils and between pupils. There is a sense in which all the talk is social: it involves social interaction between the participants. However we are interested in the contribution of the talk to the social contexts of the learning environment that the teacher and pupils are creating.

The mathematical dimension is concerned with the mathematical component of the talk and relates to the way in which the talk contributes to mathematical forms of life, particularly those that support mathematical thinking and reasoning. We are interested in the contribution of the talk to the mathematical contexts of the learning environment that the teacher and pupils are creating.

In order to develop some sort of scale along these dimensions, we would suggest that the social dimension can vary from open to closed depending on the emphasis of the utterance in terms of its contribution to the social relationships within the class. Openness on the social dimension would suggest contributing to open relationships that encourage pupils and teachers to view themselves as joint participants in the learning and teaching processes. Closedness would be linked with rigid interpretations of the participants’ involvement and force them to follow predetermined patterns of contribution to the talk. Jenni developed this scale from her observations that the social dimension of the utterances varied between open and closed.

The mathematical dimension can vary from low to high depending on the emphasis of the utterance in terms of its contribution to the mathematical contexts of the learning environment in the classroom. A high mathematical dimension would suggest that the utterance was closely linked with mathematical forms of life that take account of mathematical thinking and reasoning. A low mathematical dimension would suggest little relationship to these forms of life and might possibly reflect an instrumental understanding of mathematics.

It is clear that this identification of mathematical and social dimensions of classroom communication is not unique and other researchers have investigated their relationship. For example, research into the sociomathematical norms of classroom practice is described by Paul Cobb and Heinrich Bauersfeld (1995). The findings presented here differ from those of these researchers in that they were gathered from ‘normal’ classrooms rather than an experimental setting. As such they offer the opportunity to observe practices that are occurring with minimal researcher intervention.

Another researcher who has explored mathematics teaching in ordinary classrooms, albeit in secondary schools, is Barbara Jaworski (1994). Her findings make strong connections between the social and the mathematical dimensions of classroom events. Jaworski examined the talk in a number of lessons in which children were engaged in mathematical investigations and came to the conclusion that there were three important components to teaching and learning which comprised what she called the ‘teaching triad’. These components were management of learning, sensitivity to students and mathematical challenge. We would like to suggest that, in each of these components, social and mathematical elements play a part.

Our work is similar to that of Jaworski to the extent that it focuses on teaching and learning in ‘ordinary’ classrooms and explores the factors involved. In primary schools the main focus has tended traditionally to be on the learner and learner development rather than the mathematics that needs to be covered. This had implications for our analysis of the transcripts as it made the social as important as mathematical. However as a proviso to this the National Numeracy Strategy with its emphasis on learning objectives has possibly tended to change this focus away from the learner and towards curriculum content.

We feel that there is some gain to be made by separating the social and mathematical dimensions. This enables us to explore those strategies that are common across teaching and learning situations generally and those that are special to teaching and learning mathematics and that may
be related to mathematical forms of life. In identifying the social and mathematical dimensions of the talk we hope to disentangle some of the complex issues involved in classroom talk and present them more clearly in the contexts of primary school mathematics classrooms generally.

Methodology
As described earlier this paper presents a detailed study of talk between teachers and pupils in two mathematics lessons in primary school classrooms collected in September 2007. It is taken from a sample of observations of lessons with a small number of teachers and classes from schools situated in London. The analysis focused on the detailed study of the transcripts of these lessons, two of which have been selected as ‘telling cases’ (Mitchell 1984).

The data collected comprised audio recordings of the lessons and extensive field notes. After this a full transcription of the audio recordings was made. Analysis of this transcription identified the nature of each utterance in terms of the speaker and also its purpose using the traditional initiation, response, feedback descriptors (Edwards & Westgate, 1994). We also examined the social and mathematical dimensions of the utterances exploring the level of mathematical challenge and the opportunities afforded to the pupils to voice their mathematical thinking as well as the ‘answer’ to the teachers’ questions.

The findings
The two classes were from the same school and were parallel Year 4 groups of 27 and 28 pupils respectively. In each class the children were arranged around groups of tables with 4, 5 or 6 pupils in a group. There were two teaching assistants present in one of the classes and one in the other.

Each lesson was nominally of one hour’s length. The conduct of the lessons was tightly controlled by the teachers, though the pupils had short periods of activity during which they essentially practised the techniques taught in the more didactic part of the lesson. These periods of ‘independent’ pupil activity totalled 42 minutes out of the 2 hours (35%) and were characterised by continued question and answer between the teachers, moving round their classes, and individual pupils. Thus for some of the time the ‘independent’ working became an extension of whole class teaching though with the additional interactions between pupils.

We analysed the talk in two ways, as explained above, in terms of the social dimension and the mathematical dimension. While to some extent the classification is subjective in most cases the talk could be categorised quite clearly.

The following example of how we classified a piece of a teacher’s talk from one of the lessons will serve to illustrate the point:

…this time I am going to give you only one minute. (socially closed)

After that minute I am going to ask anyone to explain to me so I need to hear a lot of talking while you are doing your working out. (socially open)

Ok so we are going to have another subtraction, so this time 126–12. (mathematically low)

Let’s try that one. One minute. Before you start, of course, everyone listen to me. (socially closed)

The most important thing is I’m not really interested in the answer, so it’s not a race to get the answer the quickest, it’s actually how you are doing it and how are you explaining to your friends ok. that’s what the most important thing is. So you’ve got a minute to talk… (mathematically high)
The socially closed parts indicate that the teacher is very much in control and imposing restrictions on the children. However, she does encourage interaction between pupils (socially open) albeit within a much restricted context. The task set is another example of a process to be applied that has been explained and learnt earlier and as such is low in mathematical thinking. On the other hand asking for an explanation of the method used and playing down the importance of the answer is an encouragement to think mathematically and could lead to high mathematical involvement and discussion.

So, we categorised the classroom utterances within the two lessons under the same four headings. The result of this analysis is shown in the table below:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Socially Open</td>
<td>68 (34%)</td>
</tr>
<tr>
<td>Socially Closed</td>
<td>132 (66%)</td>
</tr>
<tr>
<td>Mathematically High</td>
<td>42 (14%)</td>
</tr>
<tr>
<td>Mathematically Low</td>
<td>253 (86%)</td>
</tr>
</tbody>
</table>

This result becomes more interesting if it is compared with data collected from before the advent of the NNS. Jenni (Back, 2004) found in the study for her thesis that the lesson with the highest levels of mathematical thinking and reasoning had a social and mathematical profile as follows:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Socially Open</td>
<td>30 (64%)</td>
</tr>
<tr>
<td>Socially Closed</td>
<td>17 (36%)</td>
</tr>
<tr>
<td>Mathematically High</td>
<td>150 (61%)</td>
</tr>
<tr>
<td>Mathematically Low</td>
<td>95 (39%)</td>
</tr>
</tbody>
</table>

The profile for the lesson that had the lowest levels of thinking and reasoning within her sample had quite a different profile with the dimensions tending towards the socially closed and mathematically low categories:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Socially Open</td>
<td>6 (15%)</td>
</tr>
<tr>
<td>Socially Closed</td>
<td>35 (85%)</td>
</tr>
<tr>
<td>Mathematically High</td>
<td>19 (19%)</td>
</tr>
<tr>
<td>Mathematically Low</td>
<td>80 (81%)</td>
</tr>
</tbody>
</table>

If we compare the profile for the lessons observed for this study we find that they match more closely with the second profile. The preponderance of teacher dominated talk in these classrooms has not encouraged social openness and has failed to stimulate mathematical thinking. The mathematical talk was very much of a procedural nature and lacked any mathematical challenges.

A word count of the transcriptions of the whole class teaching part of the lessons revealed that for each class the number of words uttered by the teachers amounted coincidentally to
92.6% of the total. Pupils’ utterances, making up the other 7.4%, were on average 10 words for each pupil who spoke. All but one pupil in the two classes contributed to the dialogue though many pupils’ responses consisted of one word or one number in answer to a closed question. Again we may compare this with the corresponding data from Jenni’s earlier study:

<table>
<thead>
<tr>
<th>Study</th>
<th>Proportion of total word count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current study</td>
<td>Teachers</td>
</tr>
<tr>
<td></td>
<td>92.6%</td>
</tr>
<tr>
<td>Earlier study (higher mathematical)</td>
<td>80.9%</td>
</tr>
<tr>
<td>Earlier study (lower mathematical)</td>
<td>88.7%</td>
</tr>
</tbody>
</table>

Taken together, these and the results above imply that the NNS initiatives that have been adopted by the school have not resulted in any improvement in mathematical thinking and reasoning levels. On the other hand it is clear that the NNS has had a large effect in the style of teaching. The lessons followed the pattern of an introductory mental section followed by a main part to the lesson and usually finished with a plenary session checking openly with the pupils whether they had met the criteria for success stated at the outset. There was a substantial proportion of whole class teaching and the teachers set great store by the explicit use of learning objectives (LO) and their attainment. For example in one class LO was written on the whiteboard and the children were asked if they could remember what LO stood for. Similarly, an awareness of the prescribed structure necessary for a ‘good’ lesson is shown when thirteen minutes into the lesson we hear “Now I’m going to give you one last one then we are going to start with the main lesson.” Then five minutes later again, “Well done, everybody sitting up straight ready for the main lesson.” The main lesson that ensued was similar in style and content to the introductory section but had at least been clearly delineated to the children.

Brown and Millett (Brown & Millett, 2003) have suggested that it is the quality of the whole class teaching that is the crucial factor in improving attainment in mathematics rather than whole class teaching per se. We found in our observations that the format of the lessons and much of the structure required in the NNS was being faithfully followed in schools. Our small scale study resonates emphatically with other studies, for example Smith, Hardman et al (2004) and Iannone and Cockburn (2008) and reinforces the notion that the changes taking place may be only superficial. All the indications point to an urgent need for CPD for teachers that will change the mathematical quality and openness of teacher-pupil interactions.

References

The pupils' interpretation of mathematical writings

*Communication in the teaching of mathematics is distinguished by using various systems of semiotic representation. Their usage is often given by customs or conventions in mathematics. The acquirement of rules how to form, interpret, and use them properly plays an important role in communication and cognitive processes. The problems in communication tend to be caused by not only different mathematical images or communication contexts but also non-accepting of conventional mathematical writing by the pupils or insufficient acquaintance of the pupils with rules of their usage. Some cases of pupils' inventional representations are presented and analysed.*

**Introduction**

Usage of various semiotic systems is not only characteristic for mathematics but also necessary for it enables us to communicate and cognize mathematical laws. Signs help us to grasp and understand abstract mathematical objects and relationships among them. In this context, signs mean not only mathematical symbols but rather everything used to represent (i.e. substitute) objects and relationships in mathematics, thus also words, figures, graphs, etc. In didactics of mathematics, it is common to speak of representations (although the meaning of this concept is wider). For communication in mathematical classes, it is important to acquaint the pupils not only with various forms of mathematical knowledge representation, but also with the rules how to form, interpret, and use them properly (compare Ferrari, 2006).

It is stated in curriculum documents that mathematical training leads the pupils to accurate and concise expressing themselves by using the mathematical language including symbolism, by performing analyses and writing statements in problem solving, and to perfecting their graphical expression. It may seem that pupils achieve this level of communication competences by applying the courses mentioned automatically. However, practical experience shows that if the mathematical language is not developed on all of its levels and if analyses and writing of problem solutions are performed formally only, numerous problems appear in the communication between the teacher and the pupil.

In this paper, we described some cases of usage of mathematical symbols in pupils’ writings that is not in accord with prevalent customs.

**Theoretical framework**

Bertrand (1993) indicated that the majority of pupils with problems in mathematics at the elementary school do not develop any type of representations of problems set. The pupils do not gain the impression of understanding from the teacher’s explanation but based on transformation performed by the pupil when listening to the teacher. Duval (2006) added that understanding in mathematics implies coordination of at least two semiotic representations.
The difference of images (mental representations) and contexts tends to be one of the most frequent causes of misunderstanding in the communication between the teacher and the pupil. Difference of images of the teacher and the pupil is entirely natural; it namely stems from different level of their knowledge. Therefore, it is especially up to the teacher to take this fact into account during communication, and to eliminate the misunderstanding by choosing suitable language means. Speech of the teacher should be characterized by the sociometric nature, i.e. by orientation at mental structures of the pupil (to say things in such a manner so that the pupil understands the teacher). The teacher may make understanding of the pupils and creating of adequate knowledge structures more difficult precisely by the way of communicating with them. When the teacher’s expressions are exact technically but not understandable for the pupils, mutual communication is disabled. A common language in which the concepts used have a very close content for the teacher and the pupils, and in which the words connotate similar meanings in their minds, is very important for the communication. Sometimes, a part of the language needs to be delimited, explained to each other, and agreed upon.

In communication, we rely on knowledge adopted as correct. We seek to incorporate our knowledge in the existing knowledge structure. When two pieces of knowledge are not compatible, tension is formed, and the given individual seeks to eliminate it by changing one of the pieces of knowledge or by integration of both. An important role in communication between the teacher and the pupil is played by pupils’ preconceptions, i.e. the pupils’ concepts characterized by certain immaturity, imperfection, preliminarity, provisionality. When explaining a new subject, the existing preconceptions do not disappear but rather form certain symbiosis with the new subject matter.

A significant role in words and symbols interpretation is played by the communication context, i.e. the framework within which the communication takes place. The context is determined by images of the pupil and the teacher that affect their understanding and usage of communication means, by the social environment, and cultural customs. The context plays an important role in interpreting the sign; for example, the sign “N” may denote the set of natural numbers (symbol) or vertex of a geometrical figure (index) or it can be viewed as figure symmetrical with respect to a central point (icon). And also the meaning of some mathematical concepts is given by the context in which they are used.

The phenomenon when the pupil uses an expression that does not correspond with the communication context or when the pupil uses such an expression in two different semantic contexts is called the communication confusion. When confusion in communication is caused e.g. by usage of an improper term and if the semantic context is not lost by doing so, the communication is not usually threatened markedly. Confusion caused by non-uniformity or ambiguity of the context tends to be more serious as it leads to communication dissonance if prolonged. Context non-uniformity of the communicants arises from sudden or repeated change of the context not registered by the other party or by using an expression interpreted differently by the teacher and the pupil. Communication dissonance is a phenomenon caused by communication confusion that leads to discordance or disagreement between the communicants. The source of dissonance is usually hidden and not realized by the communicants.

It is evident that the process of representation and understanding of concepts is purely individual. The need to communicate and make understood led to establishing of social conventions for the use of representation means. Representations as social conventions are expressions of intersubjectivity. Thompson (2002) defines the term intersubjectivity as the state where each participant in a socially-ongoing interaction feels assured that others involved in the interaction think pretty much as does he or she. We meet at pupils in the mathematics classroom with the conventional representation systems which are proposed to them by teacher and also with the pupils’ individual representation means, so called invention representation.
Methodology

During the semiotic approach is one of approaches used in investigation of communication in mathematics education (see Hoffmann, 2003, Presmeg, 2006, Steinbring, 2006). The main reason why the semiotic approach finds its use in didactics research relates probably to the relation between semiosis and communication. Semiotics is used as an analytic tool for the didactics of mathematics which is applicable in the cognitive, the social or the cultural level of investigation (Winsløw, 2004). In our methodology we use the triadic model of a sign: the sign is interpreted as a relationship between a sign vehicle and an object that it stands for in some way (Presmeg, 2006), and the triadic, relational view of communication (Ongstad, 2006): any communication will have a structural (form), referential (content), and an addressive (use).

During analysis of pupils’ work or statements, we seldom work with an integrated representation system. We usually deal with individual elements of such a system. This fact has led me to delimiting of the concept of the representative (Roubíček, 2006). This term denotes an element of the semiotic representation system or partial product of the representation process. Representative is a triad formed by three components: representing, represented, and representational.

The representing component or the vehicle is what represents the represented object. A mark, line, item, sound etc. can be the representation vehicle. The represented component or the object is what is represented. In mathematical training, the representation object usually means a mathematical concept. The representing component and the represented component of the representative correspond with the indicating and the indicated in de Saussure’s diadic conception of the sign.

The relationship of representation between the vehicle and the object is determined by the representational component of the representative that includes (1) a qualitative property of the vehicle identical with the object property; (2) context accompanying the representation process and delimiting the represented object; (3) impact of the vehicle-object relation on the interpreter. The type of effect of the vehicle-object relation on the interpreter is affected by the interpreter’s experience and achieved knowledge. In his triadic semiotic concept, Peirce uses the term interpretans to denote this component.

The representative is not only a key concept of the theoretic framework of the representations problems exploration but also an important methodological starting point. The semiotic analysis method is based precisely on identification of representatives and on observing of
relationships among them. Representing components of the representatives and their mutual relationships, the so-called syntax, are explored on the syntactic level of the semiotic analysis. The vehicle-object relations, i.e. meaning of the representatives, are analysed on the semantic level. Pragmatic level of the semiotic analysis is focused on exploring the representative component of the representatives, i.e. the representatives usage.

Exploring of the representatives is one of the ways how to diagnose the pupils’ understanding of mathematical concepts. Representatives that appear in the pupils’ communication when describing mathematical situations provide represent means of perceptible representation of mathematical objects. However, at the same time, they provide certain information on mental representation of these objects (i.e. on ideas created about them by the pupils). Based on these indicators, the level of the pupils’ understanding of mathematical concepts can be assumed.

**Some results**

The following text includes a summary of partial results of a number of observations of communication at mathematical classes at an elementary school (10–15 years old pupils), as well as of didactic experiments focused on the communication processes exploration.

The phenomena as a communication confusion and a communication dissonance appear in oral as well as written communication. While the communication dissonance can be eliminated by clarifying the context in a discussion or possibly by using other means of representation, in the written communication the teacher stems from the pupil’s writing, usually without having an opportunity to clarify the semantic context. Writing of problem solutions tends to be the source of numerous communication problems. For example, wrong statements of equality are seen in written solutions of calculation of an expression value using multiple operations, in which cases the pupil gradually adds parts of the expression to a partial result and uses the equal sign irrespective of the actual value of the expressions. For example:

\[
2(3 + 4) - 5 = 7 \times 2 = 14 - 5 = 9
\]

It follows from the analysis of further situations that the equal sign is a multifunctional symbol for many pupils, that denotes almost any relationship among the data. For example, the following statements appear in the pupils’ solutions:

\[
3 \text{ kg} = \text{CZK 24} \quad \ldots \text{cost}
\]
\[
1/2 = \text{CZK 250} \quad \ldots \text{be}
\]
\[
1 \text{ cm} = 2 \text{ km} \quad \ldots \text{correspond}
\]
\[
\text{cube} = 5 \text{ cm} \quad \ldots \text{have}
\]

Errors of this type the reeducation of which is usually not an easy task are caused in a certain extent by the teachers’ approach. Pupils acquainting themselves with a certain semiotic system are
not usually aware of their mistake, and it is therefore up to the teacher to correct the pupil in using the system. If not doing so, the teacher indicates to the pupil that the pupil’s usage of the system is correct which may mean improper acquisition of the semiotic system rules as a consequence.

Another problem that can be seen in solving of verbal tasks is represented by senseless usage of the unknown variable $x$ in written statements. The letter $x$ is used to denote an unknown datum, although not appearing in the calculation at all, or it denotes several different unknown variables or partial results. For example:

\[
\begin{align*}
\text{Width} & \ldots 5 \text{ m} \\
\text{Length} & \ldots 14 \text{ m} \\
\text{Area} & \ldots x \\
S & = a \times b \\
S & = 5 \times 14 = 70 \text{ m}^2 \\
\text{By 1/10 more} \\
\text{CZK} & 350 \text{ per m}^2 \\
\text{Total} & \ldots \text{CZK} \times \\
x & = 70 : 10 = 7 \text{ m}^2 \\
x & = 70 + 7 = 77 \text{ m}^2 \\
x & = 77 \times 350 = \text{CZK} 26\,950
\end{align*}
\]

The mistake can be found on part of the teachers again who have not acquainted the pupils with these forms of written statements. It must be realized that the exemplary statement or the solution course using an unknown variable, provided in the textbook, is not always the only correct one, that other possibilities exist, as well, and it is up to the teacher to acquaint the pupils with them. Verbal tasks can be solved not only by means of an equation but also by means of judgment, by listing of the elements (table), in the graphical form (graph, scheme). No unknown variable may thus appear in the writing. However, it should be apparent from the writing what and how the pupil has calculated (or how the pupil has thought) and to what solution the pupil has come.

**Conclusions**

Functional communication in the class represents an important precondition of an effective teaching process. It is apparent that it is never possible to arrive at complete agreement in communication between the teacher and the pupil for their knowledge of mathematics is different. It shows in some class situations that the pupils communicate on a given mathematical problem more easily among themselves than with the teacher. The teacher namely attends to the formal aspect of the communication while the pupils use also unconventional expressions in mutual communication, vague from the teacher’s point of view, however, understandable for the pupils. Communication in the mathematics classes is thus about seeking for a common language in a certain extent.

The occurrence of inventional representations in pupils’ mathematical writing is natural and frequent in particular at non-conforming and inventive pupils. Some of these deviations from prevalent forms of communication in the teaching of mathematics are inconsiderable, another ones present a didactical problem because they conflict with mathematical laws or procedures. In these cases it is necessary to intervene in due time for that reason the postponed rectification or reeducation is an exacting and lengthy process.
Practical experience from the classes shows that precisely the mathematical terminology and symbolism tends to represent an obstacle for some pupils in understanding mathematics. The problem does not usually consist in terms and symbols themselves but in ways in which they are introduced and used in mathematical classes. When introducing the symbolism of mathematics, it is necessary to acquaint the pupils not only with the form of individual symbols but also with the rules to create admissible combinations of the symbols, their meaning and usage in various contexts. If the pupils do not know such rules, statements written using the symbols become a formal matter for them in the better case, or a communication and cognitive obstacle in the worst.

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References
Problem posing, problem solving
Approaches to solving processes of one fuzzy problem by primary school children

We have focused on description of different pupils’ ways of solving the fuzzy problem and to monitor development of mathematical thinking depending on the children’s age. This is based on the process of solving the problem. Results of this experiment are obtained by the method of detailed analysis of pupils’ solutions.

Fuzzy problems are a new tool for math teachers. The researchers of mathematics education have not dealt with this matter so far. Therefore we have examined it in our research. We have studied the subject from different point of views. In this paper we have described the primary school pupils’ attitude to fuzzy problem. In our study we have concentrated on primary school children because their mathematical thinking is already developed at that age.

Impulses in our real life are ambiguous as well as mathematic problems. The most suitable way to clarify this subject is to explain it by using an example. This assignment was used by M. Hejný and F. Kuřina to describe and show the different ways of teaching (Hejný, Kuřina, 2001). The assignment is originally from the forth year’s schoolbook. The example of assignment of fuzzy problem:

Count all even numbers between the numbers 5 and 21.

The Slovak language enables two different meanings and understandings of the word “to count”. One is with the meaning “to sum” (it gives a result of 104) and the second meaning is “to count up = to find out the quantity” (it makes the result of 8).

In our study we used the expression “the fuzzy problem” to define a mathematic problem expressed in words whose assignment can be interpreted in different ways.

The formulation of the aims of the work

It is obvious that child’s understanding of the world around him/her (as well as mathematics) is different than the view of an adult. Therefore solutions of the pupils in the first four grades at the primary school enable us to see the fuzzy problems from different points of view.

The aim of this study is to describe different pupils’ ways of solving the fuzzy problem and to monitor a development of mathematical thinking with the age of children. This is based on the process of solving the problem. We suppose that pupils of the first four grades are not bound by a following didactic contract: “I do not say what I consider to be right but what I think the teacher thinks is right.”
Methodology

To fulfill the aims of this study we consider the qualitative research method to be the most appropriate. In the experiment we used an individual discussion with pupils and also a discussion in the group.

The experiment was carried out in two parts: in April 2007 in the second grade (three children of age 7–8 years) and in May 2007 in the fourth grade (three children of age 9–10 years) in one of the primary school in Prague. At the very beginning of the experiment we had an introductory conversation to get to know pupils better. This conversation was used as a start of cooperation between pupils and us. We told them in advance what was going to happen. We sent two of them back to the class and the third one was given the assignment on the paper that he/she should solve. After he/she had solved it the second child came back and was told the assignment by the pupil who already solved it. Then the second child solved the problem. The same situation happened when the second child told the last pupil the assignment. After the last solution was given all of them were asked to work on other four differently reformulated assignments of the same original task. They solved all of these problems.

As an example of the fuzzy problem we used an assignment with twelve cubes. The same assignment was also used in the research carried out with secondary and college students (Kovárová, 2006) so it enables us to compare the ways of solving the task by primary and secondary schoolchildren. Of course, it was necessary to make a little modification of the original task for second grade pupils because they may not know the meaning of the word “block” yet. For the fourth year pupils we can use the original task with the word “block” (expression used in fourth grade is in brackets):

How many blocks of flats (blocks) can you build from twelve cubes?

The words “block of flats” substituted the word “block” for second grade children because we do not want to define new concept. In the first part of the experiment (introductory talking) we were discussing the meaning of the word “blocks of flats” to make the idea clear (block of flats = block).

As it was mentioned above the elder students in the secondary school and the college students have solved this problem. From the output of research with the secondary school and college students we reformulated the original assignment and we got four different interpretations. These interpretations were given to the pupils at primary school in the last part of the experiment. By these reformulations we wanted to monitor the abilities to read and understand mathematical text.

The reformulations were given in this order:
1. How many different blocks of flats (blocks) can you build from twelve cubes using all cubes for each block of flats (block)?
2. What is the maximum number of identical blocks of flats (blocks) that you can build from twelve cubes using all cubes?
3. Using all twelve cubes you have built several identical blocks of flats (blocks). What different types of blocks of flats (blocks) could you build?
4. How many blocks of flats (blocks) can you build from twelve cubes if you do not have to use all twelve cubes and each time you build different block of flats (block)?

For better understanding of the assignment there are attached complete answers for each interpretation (Tab. 1). Explanation for signs used in the table: for example $6 \times 1 \times 1 \times 2$ means six blocks in dimension $1 \times 1 \times 2$. 

\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Original Task} & \textbf{Reformulation 1} & \textbf{Reformulation 2} & \textbf{Reformulation 3} \\
\hline
How many blocks of flats (blocks) can you build from twelve cubes? & How many different blocks of flats (blocks) can you build from twelve cubes using all cubes for each block of flats (block)? & What is the maximum number of identical blocks of flats (blocks) that you can build from twelve cubes using all cubes? & Using all twelve cubes you have built several identical blocks of flats (blocks). What different types of blocks of flats (blocks) could you build? \\
\hline
\end{tabular}
We supposed that it is difficult for primary schoolchildren to write down the process of solving the problem. Pupils could use cubes (more than 12, approximately 100). They showed us their solution manually so we could see their work and also the processes of thinking and solving the problem. The processes of the solving were recorded with the video camera and written down into the protocols (the notes of all that was done and said). Afterwards we deeply analyzed the protocol with the detailed analysis method. We can find this kind of analysis published in paper (Kratochvílová, Swoboda, 2003). We tried to describe all phenomena that we have observed. The phenomena were divided into three groups: social, emotional and cognitive (Kratochvílová, Swoboda, 2003). Social phenomena include all behaviour which is influenced by human interaction with each other. Emotional phenomena are mental and physiological states of a human being associated with a wide variety of feelings. Cognition phenomena are

<table>
<thead>
<tr>
<th>Quantity of used cubes</th>
<th>Interpretation No. 1</th>
<th>Interpretation No. 2</th>
<th>Interpretation No. 3</th>
<th>Interpretation No. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cube</td>
<td>12*1x1x1</td>
<td>12*1x1x1</td>
<td>1x1x1</td>
<td></td>
</tr>
<tr>
<td>2 cubes</td>
<td>6*1x1x2</td>
<td>6*1x1x2</td>
<td>1x1x2</td>
<td></td>
</tr>
<tr>
<td>3 cubes</td>
<td>4*1x1x3</td>
<td>1x1x3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 cubes</td>
<td></td>
<td>3*1x1x4</td>
<td>1x1x4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1x2x2</td>
<td></td>
</tr>
<tr>
<td>5 cubes</td>
<td></td>
<td></td>
<td>2*1x1x6</td>
<td>1x1x5</td>
</tr>
<tr>
<td>6 cubes</td>
<td></td>
<td></td>
<td></td>
<td>1x1x6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1x2x3</td>
</tr>
<tr>
<td>7 cubes</td>
<td></td>
<td></td>
<td></td>
<td>1x1x7</td>
</tr>
<tr>
<td>8 cubes</td>
<td></td>
<td></td>
<td></td>
<td>1x1x8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1x2x4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2x2x2</td>
</tr>
<tr>
<td>9 cubes</td>
<td></td>
<td></td>
<td></td>
<td>1x1x9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1x3x3</td>
</tr>
<tr>
<td>10 cubes</td>
<td></td>
<td></td>
<td></td>
<td>1x1x10</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1x2x5</td>
</tr>
<tr>
<td>11 cubes</td>
<td></td>
<td></td>
<td></td>
<td>1x1x11</td>
</tr>
<tr>
<td>12 cubes</td>
<td>1x1x12</td>
<td>1*1x1x12</td>
<td>1x1x12</td>
<td></td>
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<tr>
<td></td>
<td>1x2x6</td>
<td></td>
<td>1x2x6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1x3x4</td>
<td></td>
<td>1x3x4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2x2x3</td>
<td></td>
<td>2x2x3</td>
<td></td>
</tr>
<tr>
<td>result</td>
<td>4</td>
<td>12 or 6</td>
<td>21</td>
<td></td>
</tr>
</tbody>
</table>

Table 1
associated with the development of concepts or individual minds. In this paper we focused only on the cognitive phenomena. The emotional and social phenomena were not described in this paper but they are also important. We used these cognitive phenomena to describe pupils’ ways of thinking during the process of solving problems.

The results of the experiment
It is natural that the pupil for example solving interpretation No. 1 did not build all 4 blocks that is the correct answer. We cannot expect complete solutions from primary pupils, as they are not mathematically mature enough. We can estimate in which interpretation they are solving the assignment just according to one block of flats (block) they have built and according to their process of building this block of flats. For better understanding of the following text we attached pictures of the 3D solution. Abbreviation used below for marking is in a form of Number and Letter. The number stands for the grade (2 for second grade, 4 for fourth grade) and the letter stands for an order of the pupils (A for first pupil, B for second pupil, C for third pupil).

We analyzed pupils’ solutions with the detailed analysis method. This method is very long and takes a lot of space. For that reason we showed this method only on work of girl 2A. The other pupils’ work is mentioned only in brief comments through the paper.

The understanding of the problem by the second year pupils
We worked with a group made up of two girls and one boy:

2A

As an example we show here only a part of the protocol 2A and a part of its analysis concentrated on cognitive phenomena (Tab. 2). It is clear from the analysis, what has convinced us about the pupil’s choice of interpretation. Cognitive phenomena are described as an interior monologue of pupil (interior monologue is in inverted commas “”).

<table>
<thead>
<tr>
<th>PROTOCOL</th>
<th>2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conversation</td>
<td>Activity</td>
</tr>
<tr>
<td>2A1: Does it have to be 12 cubes exactly?</td>
<td>She has made a block 2x1x6 (she used the block 2x2x4 which she had built previously) and she went on.</td>
</tr>
<tr>
<td>Ex1: Here is the assignment; on it is written how many cubes you should use.</td>
<td>She has made a block 2x1x6 (she used the block 2x2x4 which she had built previously) and she went on.</td>
</tr>
<tr>
<td>2A2: For example, one.</td>
<td>She suddenly realised that she used too many cubes (15 cubes). She destroyed segment of block and she has made a tower 1x1x8 (Fig. 01).</td>
</tr>
<tr>
<td>Ex2: Well, so leave the block as it is and make another one.</td>
<td>She has built the same tower as before (Fig. 02). When it was 8 cubes height it broken down and it destroyed the first tower.</td>
</tr>
</tbody>
</table>
From the way she was solving the task we can see the progress of her thinking. At first she was trying to build “a tower as tall as possible”. Because of poor stability she put the two towers together. In the end she realized that she could build wider block. She considered the problem as the interpretation No. 1.

Her assignment of problem forwarding to the next girl was:

I read from the paper that from twelve cubes…, from how many cubes…. from how many cubes I can make…. How many blocks of flats can YOU build from twelve cubes?

The second girl built two blocks 2x1x3 (Fig. 05) without hesitation and she told the result: “Two”. She did not say a word about her process of solving and she did not want to build any other block. She considered solving the problem as the interpretation No. 3.
Her assignment of problem forwarding to next boy was:

I happened to know how many blocks of flats you can make from twelve cubes. You have to build more blocks.

2 C

The third boy after hearing the assignment built three blocks 2x1x2 straight away (Fig. 06). Despite the fact that they were told not to interfere with other pupils one of the girls laughed at him and shook her head in negative way. The boy broke down the buildings in anger. Later on he built a block 2x1x6 (Fig. 07).

The solution of the third pupil is interesting. He has reacted immediately to criticism and changed his solution. At first he solved the problem as the interpretation No. 3 and after the change he solved it as the interpretation No. 1.

![Figure 06](image1)

![Figure 07](image2)

In the last part of the experiment in this group there was a little disturbance because the pupils could not concentrate any longer. The pupils did not notice any differences in the four modified interpretations.

**The understanding of the problem by the fourth year pupils**

There were two boys and one girl in this group:

4 A

The first boy’s solution was surprising. He came up with new understanding of the assignment and therefore we fill up new interpretation:

5. How many different groups of blocks consisting of all 12 identical cubes exist?

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1x1x2</td>
<td>3x1x1</td>
<td>2x1x1</td>
<td>5x1x1</td>
</tr>
<tr>
<td>2</td>
<td>3x2x1</td>
<td>2x1x1</td>
<td>4x1x1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4x2x1</td>
<td>1x2x1</td>
<td>1x2x1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3x2x2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3x1x1</td>
<td>5x1x1</td>
<td>4x1x1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>7x1x1</td>
<td>5x1x1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4x1x1</td>
<td>2x1x1</td>
<td>3x1x1</td>
<td>3x1x1</td>
</tr>
<tr>
<td>8</td>
<td>4x1x1</td>
<td>5x1x1</td>
<td>3x1x1</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>5x1x1</td>
<td>5x1x1</td>
<td>2x1x1</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>5x2x1</td>
<td>1x1x2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 3**
He had built ten different groups of blocks. Tab. 3 represents exactly boy’s process of building. Fig. 08 shows what he built in the first phase, fig. 09 shows what he built in the second phase and fig. 10 shows what he built as last, creating his tenth group of blocks (in Tab. 3 in bold).

4B

The second girl built, at first, blocks 2x3x1 and 3x2x1 (Fig. 11) so she solved the problem as the interpretation No. 3.

\[ \text{Figure. 11} \]

Immediately, without any explanation, she started to solve the problem as the interpretation No. 1 (she built blocks 1x3x4 – Fig. 12, 1x6x2 – Fig. 13 and 2x3x2 – Fig. 14).

\[ \text{Figure. 12} \quad \text{Figure. 13} \quad \text{Figure. 14} \]

The girl has changed her interpretation after her questioning. Following dialogue is a part of the protocol 4B put into the text. Change of her interpretation is evident from the protocol. For example expression 4B2 demonstrate that girl didn’t feel necessity of using all of twelve cubes.

4B1: I can build 13 blocks … first build one then break it down and after build the next one.

She had built more than 13 similar blocks and then she started asking:

4B2: Is it necessary to use all the 12 cubes?
Ex1: And how many would you like to use?
4B3: Less, for example six.

After this short conversation she built blocks 1x3x2 (Fig. 15), 2x5x1 (Fig. 16) and 1x5x1 (Fig. 17). So she considered the problem as the interpretation No. 4.

\[ \text{Figure. 15} \quad \text{Figure. 16} \quad \text{Figure. 17} \]

4C

The last boy solved the problem as the interpretation No. 3. He built two blocks 3x2x1 (Fig. 18), two blocks 2x3x1 (Fig. 19), four blocks 3x1x1 (Fig. 20), six blocks 2x1x1 (Fig. 21) and one block 3x2x2 (Fig. 22) and 4x3x1 (Fig. 23).
The last part of the experiment in the fourth grade was completely different from the last part in the second grade. The fourth graders considered all four interpretations as completely different from the original assignment. Each of them was a totally new problem for them and they solved it.

The fourth graders formulated the assignment almost without change in original assignment.

**Conclusions**

It is interesting to note that 3 out of 6 pupils changed their understanding and reinterpreted their solutions. Complete change in interpretation is evident in solutions 2C and 4B. Gentle progress is in solution 2A. This is the advantage of qualitative method of research which enables us to monitor changes in strategies or interpretations of assignment.

The second year pupils did not see any differences in the four interpretations. But the fourth year pupils could read and understand mathematical text and they noticed all the differences. These results are the proof of the increased mathematical thinking of the fourth grade pupils than the second grade pupils.

**References**

Problem stories in the education for numeracy and literacy

“Problem stories” are a teaching strategy for problem posing. This work is about the analysis, through an interdisciplinary approach, of the implementation of this teaching strategy that joins Mathematics with Portuguese Language and promotes an education for numeracy and literacy.

We present and discuss some studies stressing the concern, at an international level, like the Organisation for Economic Co-Operation and Development (OECD) showed in the Programme for International Student Assessment (PISA), on the levels of numeracy and literacy of the whole population, as well as on the development of competencies. We also present and discuss definitions central to the understanding of this teaching method, particularly of problem posing and solving, and about the construction of stories.

In the first year the data was collected in a private school, from a 5 students group (group A) of eight years old average on their 3rd year of schooling. In the second year, we collected the data in a public school where the environment is a little bit problematic and poor, the group had 4 seven years old students on their 2nd year of schooling (group B). In the third year of data collection, we analyzed the group B, in their 3rd year of school, and also another group (group C), a 4 students group of eight years old average in their 3rd year of school, experiencing this method for the first time. From the analysis, we draw some conclusions reflecting on the development of numeracy and literacy competencies.

Apparently, the implementation of this interdisciplinary method, in the classroom, allows the development of numeracy and literacy competencies.

Therefore, it is possible to conclude that problem stories promote the sharing and articulation of knowledge and the development of critical sense through the cooperative work that is fulfilled.

Introduction

There are educational concerns, worldwide, expressed in institutional and research reports, that there’s a permanent need to educate for numeracy and for literacy. Those reports show that the importance given to literacy and numeracy is increasing. According to the OECD – Organisation for Economic Co-operation and Development (2003), it’s necessary to improve:

Mathematical Literacy – “An individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen.” (p.15);

Reading Literacy – “An individual’s capacity to understand, use and reflect on written texts, in order to achieve one’s goals, to develop one’s knowledge and potential to participate in society.” (p.15);
Problem Solving Skills – “An individual’s capacity to use cognitive processes to confront and resolve real, cross-disciplinary situations where the solution path is not immediately obvious and where the literacy domains or curricular areas that might be applicable are not within a single domain of mathematics, science or reading.” (p.15).

On the other hand, Bush and Fiala (1993) proposed the creation of stories as a new way of posing problems. They tried, with future teachers of elementary schools and with pupils of the 5th and 6th grades of elementary school, to apply the task of creating a story with contextualized problems.

Connecting the two together, our research problem was: “Does the creation of Problems Stories improve the development of numeracy and literacy competencies?”

Research questions

As to a definition of problem we follow the one of Krulik and Rudnick (1993) [and Posamentier and Krulick (1998)] that defines problem as a situation that students engage on, requiring a solution and, for which, the way to the solution is not known previously. We however agree with Kantowsky who points the difficulty in defining what is a problem, since one determined situation can be a problem for an individual, in a moment, and for the same individual, in another moment, only an exercise or a known fact (Kantowsky, 1977).

Bush and Fiala (1993) defend that stories with problems allow the creation of original stories and non-routine problems, and these may contribute to the logical plot of the story.

According to Brown and Walter (1990) posing problems can help students to see a topic or a pattern in a new perspective, giving them the opportunity to comprehend deeply at the same time. The authors also consider that posing problems is intimately linked to solving problems for two different reasons. First, because it’s impossible to solve a new problem without reconstructing the task. To do it we formulate new problem(s) during the whole process of resolution trying to break the original problem. Second, because after we had solved a problem, sometimes, we don’t understand fully what we have done if we don’t generate and try to analyse a new set of problems.

Lesh and Zawojewski (2007) define problem solving as a process of interpreting a situation mathematically, the authors also refer that problem solving is about seeing situations mathematically, interpreting, describing and explaining, and not simply about executing rules, procedures or skills.

As research questions, we intended to answer:

The method used in this investigation allows pupils of the 2nd and 3rd grade to build stories with problems?

The creation of stories with problems helps to educate for numeracy?

The creation of stories with problems helps to educate for literacy?

Methodology

To develop this study, we created a specific methodology, divided in three phases, and in each one, we worked to get different goals for both areas developed, always respecting the interdisciplinary approach.

In the first year the data was collected in a private school, from a 5 students group (group A) of eight years old average on their 3rd year of schooling. In the second year, we collected the data in a public school where the environment is a little bit problematic and poor, the group had 4 seven years old students on their 2nd year of schooling (group B). In the third year of data collection, we analyzed the group B, in their 3rd year of school, and also another group (group C), a 4 students group of eight years old average in their 3rd year of school, experiencing this method for the first time. From the analysis, we draw some conclusions reflecting on the development of numeracy and literacy competencies.
In the initial phase, in three sessions, we worked the resolution of non-routine problems, the expansion of problems statements and also the macro-textual analysis of stories.

The first activity proposed was the problem story “Raspel the misfortunate”. It was a story about a gnome, with a lot of bad luck, which needed to find the wish-tree to end with his misfortune. In his adventure he was always accompanied by a goat, a cabbage and a wolf. At some point of the voyage he came across a river and a problem arose, how could he cross the river to get to the tree if he only had a boat with two places. Apart from this, if they were left alone, the goat would eat the cabbage and the wolf would eat the goat. How many trips must he do to cross to the other side? We analysed the story and asked to solve the problem. In the second and third activities we posed the problems: “Riddle of St. Mathias” (When I was going to St. Mathias I met a boy with seven aunts. Each aunt had seven bags and each bag had seven cats and each cat had seven kittens. Kittens, cats, bags and aunts how many of them were going to St. Mathias?) and “Sebastian the Crab” (Sebastian the crab decided to go to the beach. He was in the sea, twenty meters off the beach. Each day, he walks four meters towards the beach. But at night, while he rests, the tide throws him back two meters. How many days will it take him to get to the beach?). We asked them to create a story including the two problems we used.

In the development phase, in two sessions, we created a version of the Snow White Story, in which she is a participant narrator. In this phase, from the traditional story, pupils had to pose coherent problems with it.

In the final phase, in two sessions, students had to create a story and formulate coherent problems with the story. This methodology was used with group A.

With group B we introduced a few changes: first we posed another problem, “The Squirrel” (A box has nine cabbage eyes. The squirrel leaves with three eyes per day, however he takes nine days to transport all the cabbage eyes. How do you explain this fact?); besides, in their second intervention, we posed new problems: “Indians paths” (The chiefs of the Indian tribes of Sioux, Oglala, Comanche, Apache, Mescaleros and Navajos gathered for a big Pow-Wow (it’s how the Indians call their meetings). In the top of the hill, they put their tents making a circle. Each tent had a path to the others. How many paths were there?) , “Jealous boyfriends” (Two couples of jealous boyfriends, John and Joana, Antony and Antonia, wanted to cross a river were there was a little boat. This boat took only two persons. The problem was that the boys were so jealous that they would not leave their girlfriends with the other boy, even if the others girlfriend was there. How could they do to cross the river?) and “The thinker” (André thought of a number. Then he multiplied it by two. After that he subtracted five. The result was thirteen. Which number was it?) and we asked again for the creation of a story including the problem we used. These last problems were used also with group C, however this group didn’t have any contact with problem stories like groups A and B.

In the final phase, in five sessions, students had to create their own story and formulate coherent problems with the story, like group A had made but in just two sessions.

Data analysis

The tools we used to collected data were videos and artefacts produced by the students, direct observations and field notes.

The data collected was analyzed according to the research questions, was given attention to possible theory that could emerge as well as the new possibilities that were not contemplated in the questions, new issues.
From this analysis, we can draw some conclusions. They felt some difficulties in the application of mathematical concepts or using mathematical competences, showing:

Absence of contact with non-routine problems – groups A and C revealed lack of contact with non-routine problems, since in their first attempt to solve the problems they referred and used arithmetic operations. The group B, maybe because they were used to solve non routine problems from the beginning of 2nd grade, they were always open-minded in the problem solving and didn’t use one of the four arithmetic operations as the only strategy, referring to drawings, schemes and charts, and including them in their problem posing;

Group A – Riddle of St. Mathias:
   Guidinha: It must be a multiplication, because they are too many.
   Vilela: Each aunt had seven cats, so seven multiplied...

Group B – Sebastian the crab:
   Bernardo: It’s easy, we draw a beach, a beach and...
   Interviewer: You do a beach and...
   Bernardo: And we put the meters. One, two, three,... We are going to draw a ruler.

Group C – Problem posing in the Problem Stories:
   "..."He took two weeks to travel across two roads. Per day he walked 467Km.
   How many Km’s he walked in two weeks?"...

Group B – Problem posing in the Problem Stories:
   "..."After fifteen minutes, the police arrived in three cars and of each car came out three policemen that arrested the three mans.
   David after ponder a while, ask his colleagues:
   In how many ways can the policemen transport the bandits?"...

Difficulty in carrying out measurements – they revealed difficulties in the application to real-live situations. These difficulties compromised problem solving because even when the reasoning and the solving strategies were correct, the solution was often wrong. Pupils didn’t considerer the unit, they counted the numbers, so for them, zero was one, and the one was two and so on;

Group A – Sebastian the crab (drawing a ruler and counting):
   Anita: So, 1, 2.
   Interviewer: It’s that way you count the units?
   Anita: 0, 1, 2.
   Interviewer: Or you count spaces 1, 2 or you count units 0, 1, 2.

Group B – Sebastian the crab (drawing a ruler and counting):
   Daniel: I’ve done it!
   Interviewer: Measure it. I want to see if it has the twenty. Count it that I want to see.
   Bernardo: 1, 2...
   Interviewer: Its starts there Daniel?
   Daniel: 1, 2...
   Interviewer: You do not understand. Show me a ruler Daniel.... Look at the ruler, where does it start?
   Daniel: In 0.

Difficulties in locating relevant information and to separate it from accessory information – students in some problems weren’t able to separate relevant from accessory information, which stopped them reaching the solution. This situation was more obvious in the riddle of St. Mathias where there was a lot of accessory information;
Group A – Riddle of St. Mathias (after a few attempts to solve the problem):

Vilela: I was going with Guidinha to St. Mathias.
Interviewer: Guidinha was going to St. Mathias and found Vilela. Vilela had seven aunts and, each aunt had seven bags, each bag had seven cats and each cat seven kittens. But at the end of all this, who was going to St. Mathias?
Vilela: Guidinha.
Interviewer: Why do you think that it’s only Guidinha?
Vilela: Because it was Guidinha who is going to St. Mathias.

Difficulty in verbalizing the reasoning used – this reveals a lack of development activities. These activities must be performed after problem solving because they allow pupils to broaden reasoning skills that can be used in future situations. In all groups, students revealed initially difficulties in describing their reasoning and the strategy used to reach the solution. They appeared to consider enough writing the obtained result.

Group A – Sebastian the crab:

Interviewer: Then you have to show me in the paper how have you done it, even if you have to make a drawing.
Vilela: I was doing 2-4-6-8.
Tobias: I was making 4 minus 2.
Interviewer: He has to walk 20 meters.
Classroom teacher: How many meters he walked?
Vilela and Guidinha: 4.
Vilela: But the sea drags him back 2.
Interviewer: But how many does he walk per day?
Vilela: 2.
Guidinha: 2 and 2 are 4.
Vilela: 2 days.
Guidinha: 4 and 2 are 6.
Vilela: 3 days. Carry on.
Guidinha: with more 2, 8. And more 2, 10.
Vilela goes counting with his fingers the days that Guidinha is counting.

This methodology promoted the use of problem solving and posing strategies. Pupils used and articulated their mathematical knowledge in a more conscientious way, which took them to initiate metacognitive actions. Every time students posed a problem they had to solve it and relate it to the plot of the story. So they had to predict the consequences of the solution to the story. Pupils became aware of the various components of a problem while formulating it, which strengthened their problem solving skills. Problem posing developed their capacities of identification and articulation of knowledge. That allowed them to develop the right strategies to take decisions as well as collaborative skills.

The construction of problem stories promoted an education for numeracy by giving students a new way of interpreting, understanding, solving and posing problems. The situations from the stories developed in students a critical attitude towards text problems and allowed them to communicate, share and explore ideas and strategies of problem solving in an effective way.

The construction of problem stories allowed educating for literacy, because it made students adopt a critical reader attitude – developing, in a conscious way, text construction processes in which the information was carefully and strategically managed in order to cause in the model reader some perlocutionary effects. This way students did learn to use language in its pragmatic
function; develop skills to understand, to use and to reflect on written texts to reach their objectives – these made students to ponder about the text by expressing their ideas, their experiences and their own thoughts.

**Final reflection**

We considerer that problem stories promote the development of numeracy and literacy competences. They demand the development of writing and interpretation skills, creative thinking and the ability of posing and solving problems. By working them in an interdisciplinary way, they develop themselves mutually and establish bridges between the disciplinary knowledge, changing positively their performance in these two different areas of knowledge.

We would like to leave a few final thoughts: first, this study allowed us to observe interesting connections that had been established between mathematics and Portuguese language, bringing up interdisciplinarity; second, the complementarities that all areas possess are a vehicle to reach new competences in the future, through an interdisciplinary approach.

**References**

The differences in mathematical thinking of children in the in-school and out-of-school situations

The report refers to the preliminary diagnosis of the difficulties that children have while adapting their mathematical skills acquired at school in real-life situations. As the primary school students and their parents stated, they rarely find school knowledge useful in real-life situations. Simultaneously, they do not try to use their out-of-school experience in math classes. The problem was also noticed while analyzing the results of the competence tests conducted annually in the sixth class of primary schools. Thus we can observe a divergence between the “school” and “life” mathematics, which results in the students’ unwillingness to learn, their lack of effectiveness and problems at tests and external exams.

Introduction

I am an English teacher, I work mostly with primary school students in Wieloglowy. I am not in charge of any class, and I take care of the Students’ Board, so the students often come to me to share their problems and ask their questions. Surprisingly often the question they ask is “What do we learn this for?”

This is a really good question: what practical use may the students make of all the theoretical knowledge they gain – often with difficulties – at school? Is there any connection between the school subjects and real life? I tried to look closer at the problem.

What do the core curriculum and teaching standards say about preparing students to dealing with real-life situations?

The core curriculum for the primary and lower secondary schools states clearly: “in school the students develop their ability to use the knowledge gathered, so that they can be fully prepared to work in conditions of the modern reality”. And more: “teachers (...) aim at giving their students the awareness of the real-life usage of each school subject, as well as the whole education at a given stage” [1].

In every curriculum of mathematical education one of the main aims of education as such is preparing the students to dealing effectively with the real-life situations.

For example, in the introduction to the “Mathematics made to measure” curriculum the authors share a very private statement: “we want our students to rise mathematical knowledge and skills, which they will find useful in their lives. (...) We also want to use the chance given by mathematics and develop their minds.” [13]

The authors of the “Aiming at sunny future” curriculum of early education go even further saying: “the fundamental aim of math education is (...) using the knowledge in practical circumstances.” [6]

We can see the concern in fulfilling this aim in the choice of taught issues. Especially the text-based tasks are considered to be the best way to prepare the students for the real-life challenges, in which they may need mathematical skills to succeed.
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The students’ and their parents’ opinions and the report of the central examination board
That is what the officials and the authors say. I asked my students and their parents to tell me, if math classes knowledge may, in their opinion, be useful out of school. These are some of their answers:
– Well, she won’t need that in the greenhouse… (Ms Urszula B., Paulina’s mother)
– If only you knew what stupid things we’ve got to learn… (Robert D., year 6)
– Miss Daria, what do we learn these equations for? (Dawid K., year 6)
These opinions prove that, in the students’ and parents’ judgments, the primary aim of education is not even close to be fulfilled. Similarly the annual evaluation of the students’ achievements shows, that the area of “practical use of gained knowledge” is the weakest point of the system. The Central Examination Board report after the exam in 2004 says: “the achievements in the area of practical use of knowledge seem to be low. The students could get 8 points and got the average of 3,95” [5]. In the next years some improvements in this area could be seen, but it is still the most difficult part of the test for the students.

Not only Polish problem
It is worth stressing that it is not only Polish primary school problem. The fact that school education is not congruent to the challenges of real life was also seen in other countries. The problem was recognized as really serious and the attempts were made to adjust school teaching to changing needs of out-of-school life. The effect of research in this area in Holland was the Realistic Math Education (RME) theory created by H. Freudenthal [9]. It is known in Poland from the beginning of the 1990s [12, 10].
But the deliberations over this matter are mainly theoretical. I did not get to any analysis or research showing the range of correlation between the school teaching and the real needs of the students. So we do not really know in which points those two trends of education meet and in which points they part. But precise assignation of those meeting and parting areas is vital, if we want to move school mathematics closer to reality and make it understandable and useful for students. This will surely cause a raise in the students’ motivation and will enable them to have better knowledge of the learning process.

Over the world
For now, I got only to one research material presenting the connections of in-school and out-of-school math learning. To be exact, it is about using a real-life materials to present mathematical issues in class. The research on that matter was held by Cincia Bonotto from Padova [2]. She tried to introduce new mathematical issues using real-life materials like bills, recipes or bus stop timetables. Her work, however, tends to concentrate more on the differences in the students’ experience, which are caused by their coming from different social or culture groups.
Other materials I got at are meant to show the advantages of the RME over the more traditional methods of teaching. Such research works were held in Holland [7], the USA [11], the UK [8], China [3], Taiwan [4] and other places. Quite a lot of them were conducted as parts of the “Mathematics Education into the 21st century” project, in which Poland also took part1. In Holland, there was even an idea of changing the teacher training in such a way, which would prepare the teachers better to using the RME.

1A conference entitled The Future of Mathematics Education was a part of The Mathematics Education into the 21st Century Project. It was held in Ciechocinek in June 2004.
Where is the problem the most intense?
The divergence between the “school” and “life” mathematics can be most obviously observed in the primary school. At the same time it is the cause of our students losing motivation to learn and their work being less effective.

The primary education is divided into two stages:
⇒ the early education, where math education is integrated with other subjects; at this stage there is almost no distance between the in-school and out-of-school learning, because of the idea of integrated teaching. It is also psychologically motivated, as for children of 7–9 years of age only the real situations can serve as a basis for education.
⇒ primary school (years 4–6), where mathematics becomes one of many subjects and gradually, because of more and more abstract issues and tasks, moves away from out-of-school reality.

As the connection between life and education is strongest in the first years of school learning, I decided to concentrate my work on the graduates from the first stage of primary school. I want to establish the touch points and the divergence points of those two streams of education.

The competence of a ten-year-old
Finishing the third year at school children are about ten years old. A child of this age is usually quite independent and the parents expect him or her to deal with quite a lot of problems like shopping, or planning nearby travels. The child’s success in such situations is often related to his or her mathematical knowledge and the ability to use it properly.

We cannot forget the significant differences in the upbringing of a child in a city or town and in a village. Children in villages are expected to be much more mature. The parents want them to take care of younger siblings, often when they are ill, to help with housework, to do everyday shopping, also in quite distant shops. Again, to succeed in such situations the child needs a lot of mathematical competence.

Examples
A lot of people in Wielogłowy have greenhouses and grow different plants in them. Most of them grow chrysanthemums. One of my students, Paulina, comes from such a family. Paulina is no school genius, for every “passable” mark (a three) she needs to work really hard, and every “good” (a four) is a holiday. From the fourth grade onwards, math has been Paulina’s horror. She has never managed to do anything right, never understood anything, never could do her homework without help. But every year, from about mid-October, you can meet Paulina in the afternoon near flower shops close to cemeteries. She is surrounded with lots of pots with chrysanthemums of each and every possible kind and accompanied by her older sister. Paulina has no slightest problem with remembering prices, counting how much the customer needs to pay or with changing money. She never mixes orders, even if somebody wants to buy a couple of flowers, different kinds, colors and sizes. Her parents told me, that she is the best flower seller of all the family.

How is it possible, that a child so brilliant in out-of-school reality has such problems with learning math? Or maybe – how come that a child with such math problems is so good at real-life math?

I decided to talk to Paulina about this. She told me, that she somehow “sees” the colour and type of flowers her customers are about to buy. She calls it “premonition” – she just knows. This feeling usually occurs to be right. When it comes to remembering the prices, Paulina uses a simple trick. She chooses the most common type of chrysanthemum and writes down its price on a rectangular piece of cardboard. On other pieces of cardboard she notes down only the distinctions from that “basis”. She puts the cardboard pieces under the flower pots and as the day passes she knows the basis price by heart and checks the differences only.
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But sometimes the problem is opposite. Darek, for instance, is a quiet, peaceful boy in year 5. He is not a genius either, but he manages to thrive quite well. In school. Out of school subjects and homeworks Darek gets lost immediately. As his mother says, she would never send him on a mission of shopping. Even if he managed to get to the right shop, which is highly improbable, he would probably buy everything but the items listed by his mother.

When I was talking to Darek’s mother I learnt that he is very shy and doesn’t like asking questions, especially to strangers. On the other hand, he has some problems with concentrating and remembering complex instructions. Probably this is the reason why the formal class language seems easier to him and clear written instructions are easier to follow.

How does it happen?

Undoubtedly, the children’s learning, also math learning happens in two parallel ways:
⇒ in everyday situations, when children help their parents, observe them, follow their instructions and so gather knowledge and abilities (like measuring or changing and counting money in shops)
⇒ in school, where suitable information and abilities are improved in unnatural, abstract situation created and guided by the teacher.

Mathematics, and quite a lot of it, is present in everyday life. For example:
⇒ illness – a child has to know how to check temperature, what medicine is to be bought and where, how much medicine is to be taken and when
⇒ nearby travels, to the shop or school – a child needs to deal with the timetable and read it correctly, needs to deal with money and tickets, needs to estimate how much time the journey will take and how much money he or she can spend in the school kiosk
⇒ everyday washing – a child needs to be able to group things of similar colors, needs to know how much washing powder to use and estimate how long the washing will take.

In all the above situations, and lots of others, the child’s success depends on how much math he or she knows and applies.

Research program and methods

I intend to concentrate my research on the group of 10-year-olds, who live outside cities and towns. First of all, I need to find out to what extend math education worked over in the early school education prepares the student for dealing with real-life situations. In order to do so, I need to concentrate on two tasks:
⇒ register the everyday situations in which a 10-year-old may need to use math skills; then isolate mathematical activities conditioning the child’s success in those situations;
⇒ identify the issues in the curriculum which directly prepare the child to dealing with real-life problems.

I will analyze the curricula, textbooks and workbooks used by the teachers and I will list all the mathematical activities necessary for a child to succeed in everyday situations. Comparing the list to the curricula and the outcome of the analysis, I will be able to find out the extent to which school math matches the children’s real needs.

At this point, a problem occurs. An analysis of the contents of education will not tell me why our students have problems with applying their knowledge in out-of-school reality.

Therefore, the second aim of my research is to explain the causes of such difficulties in applying school knowledge in real-life situations and vice versa.
In order to know this, I need to find out what mental strategies children use while facing similar tasks in school and out of it. So I shall concentrate the next stage of my research on answering three questions:

⇒ how do the children cope with real-life situations needing math skills, especially the ones who do not achieve much at the math lessons?
⇒ what strategies do they use out of school?
⇒ what strategies are used in school while dealing with texting tasks in school?

By observing children in school and by talking to them I will get to know what paths they follow dealing with texting tasks in usual lessons and tests. By talking to the parents I will learn what housework the children do, if they cope with them well and what mistakes they make. In this way I will find out what mental strategies the children use at home and at school.

Then I will compare the strategies used by children in and out of school, so I will be able to state the divergences and suggest reparation activities.

Sum-up

Considering the contents of math education, the core curriculum as well as the specific math curricula are designed to match the children’s real-life needs well. We can assume that the difficulties the students have, which occur at external tests and the divergence between “school” and “life” math that they and their parents see have a different source. The problem may rise from the fact, that in everyday situations the children build their strategies on “I know how to do it”, and in school they build them on “I know the method I should use” basis.

At the same time we need to consider the fact, that the author of textbooks and texting tasks, as well as math teachers, are adults. Their ways of thinking and coping with reality differ from those of children. They often pass over some content thinking it is obvious. For a child, a task with such “hidden assumptions” may be very difficult to understand.

In everyday life the children often try things out and change their ways if necessary. They try different methods of resolving problems and choose the best one. In school they are expected to follow the teacher’s plan, use given methods on a given level of complexity.

While at home a child knows exactly the benefits coming from doing a task correctly and punishments for not doing so, at school it not that simple. The child’s success, or lack of it, is measured by abstract marks, which may be important for teachers or parents, but do not have a practical meaning for a child.

What is more, in everyday life a child is given clear, understandable instructions. If something is not clear, a child can always ask for further guidance. At school, especially doing tests, the language of instructions is artificial, highly abstract and unclear, and there is no chance of clarifying. Coming from reality to abstract issues, although gradual, is very difficult for a young mind.

References

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The differences in mathematical thinking of children in the in-school and out-of-school situations

There is a lot of mathematics to be found in the streets. The world around us provides many opportunities to come up with 'mathematical' tasks based on everyday situations. Realistic problems lead to authentic presentations of questions to be tackled by pupils of various levels. In maths classes in school, pupils are confronted with descriptions of everyday situations. They have read and realise what is the situation in order to use their mathematical knowledge to solve those everyday situations. These descriptions of every day situations are not needed in mathematics walks, because you meet the situation and that makes the questions more obvious. Moreover, pupils are encouraged by the discovery that maths knowledge and skills actually help them to cope with many real-life problems. In mathematics walks we aim to bring pupils into daily-life situations in which they see that you can recognise, use and discover mathematics. Maths turns out to be versatile, interesting, enjoyable and sometimes surprising. Reason enough to get out of the classroom and take a walk in the school's surroundings.

Realistic mathematics education
The most important point of the mathematics walks is to learn to recognize mathematics in the world around us. As soon as something can be seen or calculated in realistic situations, it becomes clear how mathematics can be used. In mathematics walks there is plenty of scope for the use of knowledge and skills learnt in maths lessons. These have to be applied in new situations. The pupil does not know beforehand what knowledge is useful in a particular situation; therefore this can lead to a great diversity of approach. The development of individual ways of solving problems is an important objective of realistic mathematics education, and maths walks give an opportunity to practise this skill.

During realistic mathematics education meaningful tasks are formulated in a realistic context. By examining a number of different situations, the pupil discovers what is always 'the same' in these situations. In this way an abstract concept develops from the reality and is understood. Although the mathematics walks offer a great variety of contexts, it is not sensible to develop new concepts from it (on the street). However, after a mathematics walk it is possible to go back to the experience in a lesson. It is then quite possible to use the real contexts again to develop new mathematical concepts. For instance, counting the number of people going in and out of a shop can forecast the number of visitors for this shop during one week, month or year.

Using realistic contexts
Many mathematics teachers make use of realistic contexts to make mathematical concepts as comprehensible as possible. This is sometimes best done by placing the pupils in situations they can recognise. After all real-life situations appeal more to the imagination, so that abstract
Designing mathematics walks

Concepts gain meaning and are better absorbed. Realistic contexts are then deployed as examples. There are, however, other possibilities. A context with questions in which problems can be solved by mathematical means shows how mathematics can be applied. Contexts may also be used for their value in motivation, to show that mathematical skills are useful for making decisions in daily life. In short, in today’s mathematics teaching, realistic contexts are incorporated for a variety of reasons.

Presentation of contexts in the classroom

The presentation in the classroom of a realistic context is generally provided by the authors of the book. They take a great deal of trouble to make the situation as clear as useful. After all, pupils must be able to envisage the situation to be able to decide 1) what information is relevant, 2) what information is needed to answer the question, or even 3) what question you could ask in such a situation which could be solved mathematically. Obviously the teacher can use other means to make the reality appeal as vividly as possible to the imagination of the pupils in the classroom. An (imaginary) story, a photograph, drawing or video could also lead to the development of mathematical problems. In addition material can also form the context from which mathematical questions appear. For the purpose to bring a realistic situation into the classroom it have to be described and if possible supported by visual evidence. There is, however, always the drawback that the description of a situation already provides a selection of the relevant information offered by the practical situation.

Contexts during a mathematics walk

The advantage of a realistic situation in a mathematics walk is that the situation virtually presents itself. What questions can be asked? What is relevant here? Can the question be answered easily and quickly here, or is it important to go to work accurately and with precision? What are the possible ways of solving it? Such questions don’t rise separately but more or less they force themselves on you. Pupils have to find relevant measurements and data for calculations for themselves. That makes the application of mathematics more authentic.

References

Mathematics through investigations and problem solving

The paper target is to present an idea how mathematical thinking can be developed through investigations and problem solving and still covering traditional curricula. Good problem solver need to be willing to explore, to be persistent and tolerant to certain amount of frustration. For these characteristics learning environment need to be safe and simulative what would be also discussed.

A picture of school mathematics
Mathematics teaching is usually didactically and pedagogically alike to teachers’ personal experiences as mathematics learners. The content structure of teaching at elementary level is still deductive on certain indirect way, for example in geometry starting with concepts (point, line, segment …), continuing with properties and consequences. Teaching does not start with ‘all’ shapes but it starts with triangles, quadrilaterals etc. Traditional teaching is also influenced by textbooks content structure. Linear, bottom up content development seems to be very safe teaching approach. Investigations and problem solving are detached from usual maths teaching, realized only occasionally. Widely spread beliefs are that problem solving is only for gifted students, and that mathematical content is more important than teaching student to think, solve and investigate problems. Students and teachers share the conviction that problems need to be solved quickly and by that reason students need to solve before similar problems which amount depends on their ability. Solving 10 problems in one lesson is much better than one problem, what causes typical conclusion that investigations and problem solving are time consuming activities. It also happens that problems are correlated with word problems and an exercise is meant as a problem if it is complex, although with linear step by step solving procedure.

Further consideration
What we need to do that investigations and problem solving become a part of every day maths class? One of the answers is changing teachers’ beliefs what is long term task as problem solving skills and successful investigative learning environment. The second answer is that mathematical university education needs to be redefined at least for prospect maths teachers.

A seminar gives to teachers only some aspects of this complex didactical situation. Seminars combined with projects are better solution, if well planned work in the class includes appropriate evaluation. It is important that teachers understand and appreciate that while investigate mathematics students still learn mathematics even though it is unlike anything that they experienced when they were at school. For a teacher, the first step to investigative approach to teaching mathematics is to feel like a learner in given starting situation while trying to understand stated problem and looking for appropriate strategy which gives the solution. For this task we need to
choose a not routine problem for teachers. When teachers change their role from learners to teachers and they compare the content of the problem with curricula, they would like to find a content connection with it. If they can not find it some of them doubt that thinking processes and strategies are valuable target achievements for their students, because processes are long-term and difficult to assess. It is promising if we can find for seminars an unknown starting situation to teachers which still fit maths curricula and can be confidently used in their classes.

Planning investigations and problem solving is an important step. How to get an idea for a starting point? One possibility is to use the starting situations from the handbooks, articles and seminars. Teachers prefer to develop their own ideas what also shows their personal understanding of problem solving and investigations.

Usually problems and investigations are at the end of certain chapters – in the role of synthesis. Investigations and problems can be also starting points for developing new concepts.

Teacher’s role in the mathematics classroom when students are exploring is very different to that in normal lessons. Teacher becomes a listener, a fellow researcher and a questioner. What kind of questions we can use during certain process of exportation?

References

Information technology as an educational environment
Development of abstract thinking in mathematics in computer environment

This paper is summarizing ongoing research into the use of computer games as a tool for developing abstract thinking in a mathematics classroom. We will discuss the results of three educational gaming environments supplementing mathematics instruction for teenagers that were designed and student tested not only for improvement of content knowledge, but also for innovative thinking and abstraction development. The study shows that a carefully chosen digital learning environment stimulates a higher-level thinking in the majority of players.

Introduction

The commercial development of technology in the last three decades improved our everyday lives and made computerized gadgets more accessible and user friendly. As a consequence, digital games have become highly interesting to educators and researchers since their sophistication has improved considerably, not to mention that even the youngest students are familiar with computers [2, 5, 7]. We all agree that computer technology has unique capabilities for enhancing teaching and learning on every level and subject matter. Moreover, many commercial games and web sites, especially those geared toward younger kids, are already popular among parents and teachers as they address needs to master certain skills or content such as reading and counting [9, 16]. Studies indicate that computer based algorithmic repetitious activities have educational benefits and are common in American classrooms. Some games also support explorations, complexity and various contexts, including social, historical, economical and futuristic environments [8, 12, 14]. In this paper we address the need for developing games which would emphasize the development of abstract thinking [19] supporting mathematics curricula especially for students ages 12–19. We will also show three recently designed examples, went through initial testing stage as educational tools, and impacted student learning. Specific cases of games that promote higher level thinking were treated in some recent research [2, 6, 15], however due to the fact that software development is time consuming and costly, we do not have a common set of useful activities that are widely used.

The aim of the research

Over the last two years, with the cooperation of EGRIS, we designed three prototype computer games that support mathematical abstraction. The activities inspire independent thinking, creative thought through pattern recognition, categorization, modeling, strategy development, analysing and conjecturing using logical rules and previously accumulated mathematical knowledge. Our preliminary research supported by a small pilot laboratory study shows that a proper digital environment can support independent explorations and inspire higher-level thinking.
Note that our games are significantly different from games targeting language skills as humanists consider language hierarchy (of parenthesis) as their primary structure. We, on the other hand, use the time, the space and geometric and algebraic structures as basic concepts, and stress logical inferences, data analysis, mathematical modeling and proving as methodology. While the choice of the social context for the learning environment is important and makes the activities more interesting, higher level thinking skills (especially in mathematics) can be developed in context free abstract situations and be even context free.

The methodology

Since this is an early stage of our research project (especially in terms of the game development), we had 10 randomly selected students aged 12–19 play each game individually for several hours each. They were being observed, and their strategies and thinking skills evaluated by the researchers. Post survey of the activities evaluating their perception of their development skills and learning was administered. The descriptions of the three games/activities addressing different mathematical thinking skills are provided below and the results are summarized later in this paper.

1. Ace Altro - is a crime drama logic game, where the ultimate goal is to solve a mystery, ‘Who done it?’ A player/detective can evaluate statements of witnesses that are given in everyday language describing what they know about the crime. Using symbolic logic the player translates the statements into logical sentences and evaluates the outcome against the collected evidence and the set of suspects. The complexity of the statements, suspects’ diversity and evidence varies depending on the age/level of the player. Witnesses may have speaking patterns (for example giving only ‘or’ or ‘and’ statements) that can be noticed by going through different cases, so the long term success of the detective depends on the development of questioning strategy and the ability to categorize the set of suspects.

This game supplements instructions in logic: false, true statements, truth tables, logical inferences, operations on sets, categorization, conjecture making, argumentation, proofs, etc.

Example. Introductory activity: Who broke the window?

Suspects – four boys with the following characterization:
Boy A (blond hair, green shirt, black pants)
Boy B (blond hair, blue shirt, brown pants)
Boy C (black hair, white shirt, black pants)
Boy D (black hair, green shirt, green pants)

First witness: I have seen the boy who broke the window from far away. My eyes are not so good, but I am sure his shirt was not blue.

Translation into logic:
p: The shirt was blue.
True statement:
Not p

True only for boys A, C, D. So we can eliminate B from the suspect pool.

Second witness: His hair was not black or his pants were not black.

Translation into logic:
q: The hair was black.
r: The pants were black.
True statements by the second witnesses:
Not q or not r

True for boys A and D, since C has black hair and black pants.

Do you need more witnesses? Yes.
Third witness: If his hair was black, his pants were black.

\[ q \implies r \]

True only for boy A, even though boy D has black hair his pants are not black (his pants are green, so the implication is false).

Even in this introductory scenario the student has built the logical structures, analyzed them, and applied them to categorize and re-categorize elements of the set (of suspects) into groups depending on consecutive testimonies. As the cases become more complex the witness interviewing strategies play an important role in reaching the solution.

2. MonsterSort – is a combinatorial game that asks students to develop their own sorting strategies and compete with each other, or the computer character-sorting monster Gork. Several parameters are being evaluated such as random initial distribution of the numbered boxes, as well as memory requirements, number of steps, complexity, etc. Students have full access to the data, and can make their decisions to maximize the chances of success in the scientific way. This game supplements school curricula in combinatorics, probability, algorithm development, and decision-making strategies.

**Simple example.** Put boxes numbered 5, 4, 1, 2, 3 in increasing order.

Possible solution: move 5 to the side (store in the memory). Put 1 in the first spot, now the third spot is empty. Move 3 to the third spot. Move 5 to the last spot. Put 4 to the side, move 2 to the second spot, and move 4 from storage.

- Number of steps: 7
- Memory (storage) use: 1 unit

Second possible solution: move 4 and 5 to the side (store in the memory). Put 1 in the first spot, 2 in the second spot and 3 in the third spot. Move 4 and 5 to fourth and fifth spots respectively.

- Number of steps: 7
- Memory (storage) use: 2 units

Depending on initial distribution of numbers, the choice parameters, such as storage costs, number of steps costs, time, and different combinatorial solutions may optimize the best solution function, and the student will develop different strategies. The game may be played against the computer, that uses standard assorting algorithms such as, QuickSort (minimizes memory space usages), MargerSort (minimizes time), HipSort (minimizes complexity function) [see 11]. The higher levels of the game will lead students to algorithm design questions.

3. Rock, Paper, Scissors – this game is based on the well-known kids’ game, where with the uniform probability of 1/3, two players pick simultaneously a rock, paper or scissors. Rock wins over scissors, paper wins over rock, and scissors win over paper. If both players pick the same object, there is a tie. The game becomes more interesting when the probabilities of each choice are independently controlled, i.e. can be changed. Then the decision-making strategy becomes and interesting issue for the players as the environment varies. The game supports probability and statistics curriculum, data analysis, decision trees, strategic choices, etc.

Example 1: Classical case: both players pick their object with a probability of 1/3. Long-term prediction is that there will be a tie, which can be supported by data collection and analysis. So this is a fair game.

Example 2: One object case: One of the players always chooses one object, say scissors, the other stays with 1/3 strategy. Who wins in the long run?
Example 3: Different probabilities: One of the players (for example the computer) chooses his probabilities. Can the second player (the student) choose his probabilities to beat the first? Explorations, data analysis and decision trees help students choose the right strategies.

Note. These games have no specified winning strategy rather they support independent thinking, making decisions based on evidence, cooperation, discussion, and social interactions. They are set in a friendly, non-judgmental and humorous setting, so students get easily engaged and enjoy playing.

The results and their analysis
All students became deeply involved in the decision-making and the role-playing aspects of activities. Our games have no specified ‘right solution’ and this stimulated students to freely make decisions. After the initial period of random choices, they started to use collected data, experiments and conjecture testing to improve their results. Since the social setting of the project was supporting cooperation and discussions were encouraged, all students started to analyze the evidence and make categorized decisions while building various models for successful strategies. In surveys, students positively commented on the interactive nature of the games and the learning process itself. They liked the simulations and graphics, and praised instant progress feedback that positively impacted their outcomes and thinking processes. They also cited increased self-confidence in analyzing various situations using mathematics.

These results are quite impressive, as all students showed ability to develop abstract thinking skills and have learned game specific mathematical ideas. So far, we have conducted all our experiments in the small laboratory setting and the hope is that the statistical data will be similar for classrooms, small groups and homework assignments.

Specific Comments about the Games:

Ace Altro - the logic game. Most of the students did very well at the introductory levels of the game and quickly learned symbolic logic language set-up and making inferences from the data analysis. With increased difficulty, they developed various strategies to deal with data and decision-making. Most of them organized and reorganized data sets, and tried to eliminate suspects and optimize witness-questioning strategies. Since this game is cartoon-like, and there is not much interaction, it was hard to keep the younger students focused on their tasks for longer than an hour per sitting.

MonsterSort – this combinatorial/sorting game has interesting three-dimensional graphics, and students enjoyed giving orders to and competing with the Gork character. They all developed their own sorting strategies and their own sorting algorithms trying to maximize their chances to win. Initially students found the sorting questions interesting, but after several hours they wanted more variety. To make the tasks more interesting it is our plan to design various additional combinatorial problems that lead to algorithm development.

Rock, Paper, Scissors – students were familiar with the game and used the uniform probability of 1/3 for fun and to get familiar with the environment. When the probabilities were changed they got very involved in decision-making strategies while competing with each other. They carefully analyzed datasets to improve their odds of winning. Three of them made general conjectures, and tried to test them. All showed improved understanding of statistical data analysis.

Conclusions
The preliminary results of this project are very positive as all students did data analysis and were stimulated to creative mathematical thinking on an appropriate level. Various techniques
were developed and used including data categorization, modeling, strategizing for decision making, conjecturing, arguing and proving that their statements are true. Logical rules become the common communication platform amongst the players, and optimization issues become the central element of conversations. To build the connections with their previous mathematical knowledge they tried mapping and remapping structures and models they had built. We will further develop the prototypes and make them available to the larger learning community. These games are designed to support an existing pre-college curriculum and engage students in activities requiring abstract mathematical thinking.

References

Teaching computer-assisted mathematics to primary students

The subject of the research report is the natural pedagogical experiment which was to show an innovative didactic situation and changes in arithmetic knowledge and skills learnt by first grade elementary students.

The research aim

The cognitive aim of the empirical diagnostic and verifying study was to estimate the effectiveness of using computers, together with intentionally and carefully chosen computer programs, in developing mathematical skills and active arithmetic knowledge construction of first grade elementary students.

The practical implementation aim was to formulate conclusions and implications for elementary teachers and primary school headmasters, useful in adding computers into the teaching aids range helpful in developing mathematical skills.

References to significant literature and publications

Constructivism is nowadays the most significant educational trend, which describes the process of school learning, its dynamics and the relationship between the activities of the teacher and the student.

According to the constructivist approach, the student should be an active and creative action subject because it is him who constructs his own knowledge. The teacher is the one who supports the student in his activities and development. The support is based on aware creation of optimum conditions encouraging the student self-action incorporated in the process of shaping attitudes, skills and constructing own knowledge.

The great role of the child’s self action was stressed, among others, by J.S. Bruner [1], J. Piaget [2], L.S. Vygotsky [3]. New (constructivist) schools refer to real activity and spontaneous work resulting from the student’s personal need and interest. Active education underlines the fact that students should express their will to do what they are involved in. The need and the interest that results from it is a factor that produces the reaction of authentic activity [4].

Methodology

The main research question was formulated as follows: In what way does using educational computer programs in first grade elementary education affect the level of students’ arithmetic knowledge and skills? The main question was complemented by more detailed questions. One of them was as follows: In what way does computer-assisted mathematics teaching to first grade students of primary school help in developing the ability to use the knowledge in problem situations?
The global dependent variable was the level of students’ knowledge and skills in the area of elementary mathematics teaching, whereas the global independent variable was the methodology of using carefully chosen educational computer programs assisting elementary mathematics teaching.

The dominant research method was the pedagogical experiment carried out with the technique of parallel groups. The experiment went according to John S. Mill’s canon [5]. It took the form of the natural experiment, which means that the students did not know they were the subjects, and the research was carried out in conditions typical of first grade education.

The complementing research methods were: diagnostic soundings carried out with primary school headmasters and elementary education teachers, skills tests for students of both groups, observing students while learning, the documents analysis and the dialogue method. All the research tools were verified in pilot research.

The experiment was conducted partially in the first (pre-tests) and the second term of school year 2003/2004 in two purposely chosen schools situated in the Silesian province. The schools were nominated due to diagnostic soundings and the approval of the head and teachers to carry out the research.

The post-tests were carried out in June 2004, whereas in the second week of September 2004 distance research was done. Altogether 72 students of four first grade classes of primary school took part in the research, including 37 students of two classes making the experimental group and 35 students of the other two classes making the control group. The statistic analysis of pre-test results revealed that characteristic features of the students, both in the experimental and control group, were of not great difference, and that means the groups came from the same general population and they could participate in the research.

The pedagogical experiment was preceded by thorough analysis of curriculum material in the area of arithmetic, planned for the first grade of primary school in the second term, because the experiment was intended to be carried out exactly during that term. Afterwards, the computer programs’ contents were analysed and regarded as positive by the teachers in the diagnostic soundings [6]. The educational material mentioned above also influenced the choice of the computer programs. Besides the technical and didactic quality of the programs and the possibility to use them were judged on the basis of educational computer programs criteria [7]. Then the lesson plans and the schedule of the experimental activities were prepared.

Data collection tools meant to measure the level of mathematical knowledge and skills of first grade students in the area of arithmetic were also prepared. The tools were based on standardized tests by A. Cheba and A. Andrzejewska [8], verified in the pilot research, and then used in the pre-experimental, post-experimental and distance research.

Mathematical Skills Test contained 24 exercises in the area of arithmetic. It was divided into three parts (natural numbers from 0 to 10, from 11 to 20 and from 0 to 100 – denominations of ten and the complete number range), according to the stages of numbers learning [9] and curriculum material in arithmetic planned for the first grade of primary school in the second term. In the Mathematical Skills Test the students could score from 0 to 2 points for each exercise, apart from exercises 10 and 20 (in which you could score from 0 to 4 points). The level of arithmetic skills achieved by students was estimated as follows: a very high level (A), a high level (B), a medium level (C), a low level (D) and a very low level (E). All students of a particular class were tested simultaneously, in conditions ensuring that every student solved the test individually. The test took 1 hour 30 min. with one ten-minute break excluded from the overall time of the test. The Mathematical Skills Test included the following arithmetic skills: the knowledge of natural numbers in ordinal aspect; the ability to count the missing numbers in addition and subtraction.
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equations, the ability to apply number properties and symbols to make addition and subtraction equations i.e. the ability to make addition and subtraction equations using the numbers given and writing the equations with ‘+’ and ‘–’; the ability to compare the results of addition and subtraction; the ability to compare natural numbers; the ability to add and subtract natural numbers within and outside denominations of ten; the ability to add and subtract heard natural numbers in memory; the ability to distinguish units and tens in one- and two-digit numbers.

The ability test on applying addition and subtraction in problem solving was made of five word problems. In the test the student could score from 0 to 4 points for each exercise. The levels of applying arithmetic skills in problem solving were estimated in the same way as in the natural numbers equations. The test took 45 minutes. Every student completed the exercises individually, whereas all the students of a particular class were tested simultaneously. The test aimed to estimate computer influence on the ability level of applying the arithmetic knowledge and skills in solving simple and complex word problems.

During the following classes, done according to the lesson plans, the students practised and consolidated the previously introduced arithmetic knowledge and skills meant for the first grade of primary school in the second term. However, the classes in the experimental group were done in the computer laboratory using three selected programs, i.e. *Click teaches to count in the green school* (WSiP program, 1999, recommended by the Ministry of Education), *Virtual School. Mathematics* (YDP program, Interactive Publications, recommended by the Ministry of Education) and *Mathematics. Addition and subtraction* (Aidem Media program). The control group practised and consolidated the same arithmetic material as the experimental group, but using selected textbook publications [10].

**Data presentation and analysis**

Figures 1–3 present the achievements of students in the control and experimental groups measured in the pre-experimental, post-experimental and distance research. Figure 1 shows that the students in the experimental group made considerable progress during the experimental classes in computer-assisted mathematical skills teaching and constructing arithmetic knowledge. In the post-experimental test 73% of the students reached a very high level of mathematical knowledge and skills (A), 14% high level (B), 8% medium level (C) and only 5% low level (D). Among the students who achieved a very high level there were those who got the maximum number of points (50).
Figure 2 shows increase in mathematical knowledge and skills of the students in the control group. Almost all the students in the control group showed a considerable increase in arithmetic knowledge and skills, however their results in the post-experimental test are lower than those of the students in the experimental group. Only 26% of them reached a very high level (A), 43% a high level (B), 17% medium level (C), 11% low level (D) and 3% very low level (E).

Figure 3 presents the overall results of Mathematical Skills Test on numbers 0–100, used in every stage of the research. The proportional data shown in the figure enable you to compare the experimental group students’ achievements with the achievements of the students in the control group as far as all the research stages are concerned. The levels of arithmetic skills reached by the students were marked as follows: very high level (A), high level (B), medium (C), low (D) and very low level (E).

In order to check the difference significance between the two groups a Chi-square test was applied ($\chi^2$) [11]. The students in both groups progressed in arithmetic skills in numbers 0–100, whereas the differences in the pre- and post-experimental research in the development were statistically significant both in the experimental and in the control group. However, the progress
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turned out to be greater in the experimental group than in the control group, and the difference between the groups was statistically significant. The tendency was also confirmed in the distance research. The experiment confirmed the veracity of the major hypothesis that assumed that using educational computer programs in the first grade of elementary education would result in high level of arithmetic knowledge and skills achieved by the students. The major hypothesis was estimated with the 0.99 probability; for the significance level \( \alpha = 0.01 \).

The research also confirmed the assumption that computer assisted mathematics teaching in the area of arithmetic has a great impact on the development of the ability to use the learnt notions and arithmetic operations in solving simple and complex word problems (see figure 4). The levels of applying the arithmetic skills in problem solving are marked like in the Mathematical Skills Test.

The difference in the test results between the experimental and the control group in the two research stages, i.e. in the post-experimental test and distance research was statistically significant. The students learning in a traditional way achieved poorer results that the students learning with the help of computers. With the 0.99 probability (for the significance level \( \alpha = 0.01 \)) a hypothesis which assumes that there are statistically significant differences between the groups in applying addition and subtraction in problem solving was suggested.

Conclusions

The proper development of mathematical thinking demands the right organisation of the student’s activity using well-chosen teaching aids and materials, among which the computer with educational software appears more and more frequently. The results of the empirical research described above allows to suggest computer-assisted mathematical skills teaching as an integral part of education in the first, second and third grade of elementary education. Well-thought, planned and deliberate using of the new interactive teaching aid that the computer constitutes promotes considerable development of mathematical skills. Besides, it has a great impact not only on the cognitive zone, but also on the emotional and motivational one.

The right course of education based on the idea of constructivism is bound to the necessity to use wide specific material. A computer program including specific material, can make the teacher’s work easier, save not only time but also the energy and effort put into preparing specific material for every student in the class. Apart from that, the computer with substantively well composed educational programs and methodologically well used in the process of learning and teaching constitutes an attractive and student-friendly teaching aid [12].
An educational computer program that is well constructed and intentionally used to assist the development of elementary students’ mathematical skills has a positive impact on rising interest in mathematics. It creates new possibilities to increase the level of students’ engagement in achieving learning and teaching goals. Computer assisted learning becomes more attractive, draws students’ attention and changes the relationship between the teacher and students in a positive way. It creates many opportunities to construct knowledge and practise students’ mathematical skills.

Success in using computers in mathematics teaching also depends on the level of integrating them with teaching contents and methods, as well as with other teaching aids and materials. It is necessary to use them according to a plan, deliberately and systematically in every aspect of education in order to achieve a positive result in mathematics teaching. That is why the role of the teacher is to combine the traditional way of instruction with computer assisted teaching. On the other hand, the role of textbook authors is to construct educational computer programs which would complement elementary education textbooks, and to include computers and educational software in textbooks by annotations referring to computer assisted learning and teaching.

Primary school head teachers, helped by parents, elementary education teachers and local authorities should take action to provide schools not only with modern computers but also the best educational software. They should open new computer classrooms or adapt the existing ones for computer assisted elementary education and compensation classes.

Elementary teachers should be encouraged to fully exploit the educational value of the computer, and broadly speaking of information technology. It seems necessary to organise methodology workshops in which they would be familiarised with the available educational software, useful in assisting the introduced and/or practised mathematical knowledge and skills, and with computer assisted teaching methodology.

References

Procedural and paving way of building-up of geometrical object in concepts and strategies of primary school pupils

This presentation concerns geometry perceptions of primary school pupils in computer applications involving turtle graphics. The main goal was to analyze which strategies children use by discovering which part of a given shape they are capable of drawing by applying an algorithm built from commands of several given types. The application called Obkreslovačka (Tracer) has been created for assessment in schools. Pupils’ strategies and procedures were analyzed, with particular attention to how they divide a figure into parts within their turtle procedures, and how strategies and procedures depended on the specific geometrical shapes involved. This presentation reports on a set of tests of about 11-years-old pupils in 28 South Bohemian schools.

Figure 1. The environment of the testing application of Tracer. The turtle is creating the procedural part of the given figure according to the programme

Introduction
Turtle graphics contributes opportunities for modelling and drawing to the teaching and learning of geometry, both for pupil activities and for the expansion of the teacher’s repertoire of tasks. This leads to the more precise pupil conceptions. Our concern is how primary pupils are able to divide geometrical figure in two parts: a ‘procedural parts’ inscribed into the given figure and created through a procedural process (in terms of turtle graphics commands); and a part paved with given basic tile objects (typically a remainder of the given figure). In what ways do pupils discover those parts of a figure that can be drawn with the aid of turtle procedures, and
with which methods do pupils discover this shape in each specific task? What strategies do pupils employ to solve such tasks?

We have started from the requirements of Czech and Slovak experts who declare that geometry teaching in primary school should be based on the pupils’ understanding which is developed through various problem-solving experiences. For example, František Kuřína says: “Geometry in primary school has not even a preparatory deductive character, but it is markedly operational.” (Hejný and Kuřína, 2001) Hejný argues similarly: “Experimentation is a basic and irreplaceable way to obtain geometrical experience. Pupils in primary schools have a natural ambition to investigate the world by their own activity” (Hejný et al., 2004). Kuřína worked out a theoretical analysis of primary pupil geometrical experience and suggested four didactical principles that should form the didactical structure for geometry teaching in primary schools (Kuřína, 2001). The presence of the following four principles is evident in the described investigative project.

- **Space Partition** is applied when introducing the new notions into geometry (e.g., a line divides a plane into two parts). Pupils meet the space partition principle as early as babyhood. The space partition principle is included in that part of the project where a given figure is divided into parts according to the way a pupil chooses to create it, i.e. into the part that can be modelled by the use of the set of turtle graphics commands, and into the part that can be paved by predetermined basic tile objects.

- **Space Filling.** The fact that the figure contains other points of the space in addition to its sides is emphasized by the space filling principle. The areas of geometrical figures can be measured by filling the space. Calculations traditionally form a part of geometry teaching, but it is also important to teach the pupils to observe and to guess. Although the testing tasks require a pupil to trace a given figure, the pupil fills the shape with objects created by procedure or paves the shape with tile objects.

- **Constructions** mean not only solving the geometrical tasks by means of drawing but also various ways of representing geometrical figures (for example, constructing a building from two, three, or four bricks of the children’s building set). Modelling the figure through writing the programme brings the possibility of a deeper understanding of geometrical figure attributes. Above all, the use of a cycle within the programme leads to the idea of symmetry and its discovery in both shapes of parts of objects and in the shapes of their perimeter lines.

- **Movement in the space.** The idea of the direct movement and experience with direct movement is typically well-developed by a child of six years. It can be observed during the tasks in which we search the traces on the plan. The movement in the space can be interpreted as the imagination of turtle movement when tracing the geometrical figure. Here the procedural construction of a figure means the path created with turtle commands (Fig 2).
We distinguish the procedural part of the figure from the algorithm for tracing it. The procedural part depends on the set of used drawing commands, but it has only one shape for given set of commands. There are several ways how to trace procedural part and several ways how to describe the chosen trace by the turtle programme.

**Main research objectives**

Aspects of the process of conceptual understanding of polygons

- Does the use of the computer and turtle graphics affect conceptual understanding of polygons?
- Understanding of the polygon as a part of the plane (it is not an outline demarcating it, but it includes also points within the plane itself)
- Propaedeutics of the symmetry conception as one of the polygon attributes

Geometrical shape from a dynamical point of view

- How a procedural part of a given polygon will be found (i.e. the ability to find an inscribed part of the given figure, which pupils are able to model by means of given set of motion commands)?
- Which shapes seem to be simple, which difficult to children, and why?
- Which strategies do children typically employ in discovering a procedurally created shape?

**The methodology**

An application called Obkreslovačka (*Tracer*) for researching pupils in schools has been created in the Imagine Logo environment (Blaho and Kalaš, 2001). The commands of turtle painting have been limited to only STEP (go ahead a constant distance), LEFT and RIGHT (turn 90 degrees), because these primary school pupils are not yet familiar with the concept of angle. Some steering commands were added for creating a cycle: REPEAT, and brackets for defining lists of repeated commands. So the limited commands only allowed tracing the rectangular shapes with vertices which lay on a base grid and with horizontal or vertical lines. When the turtle finished the drawing at the start point, the *Tracer* filled up the drawn shape (compare Figures 1 and 2).

Assigned tasks challenged pupils to trace a given geometrical shape in the *Tracer* environment. That meant to inscribe the given figure with a maximal connected polygon of a suitable kind created by a “turtle programme” procedure. Some given figures contained skewed line segments and it was impossible to fill up the whole shape using only allowed drawing commands, so the user could add some tile objects to the figure to fill gaps in the given figure. All these tiles had the shape of right isosceles triangle and the same size. They could be dragged and rotated in steps of 90° by right click (Figure 2). The pupils were prevented from paving the whole shape with these tiles (it was impossible to fill up a basic square of the grid only with tiles). The turtle executed the programme on request of the user.

The testing tasks were divided into four groups of varying difficulty, based on the given objects and the rules for tracing:

- The lowest level included only figures of simple shape made of base grid squares, e.g. rectangles or shapes of L letter of different size and different placing of start point. This level served mainly as a means of familiarization with the application environment and the type of testing tasks.
- The shape used in the second level involved horizontal, vertical and skewed sides with a slant angle of 45 degrees (Figure 3), so that it was necessary to add tile objects to the procedural part in order to fill in the whole shape.
The third level used very similar shapes as the second level, but required different working processes. Children had to complete the programme first and after confirming their programme they were not allowed to edit the programme during paving. This level was pivotal and the most important for the research because children were forced to imagine the procedural part without the initial help of manipulation with tiles.

Figure 3. Shapes used in second and third level with procedural parts marked

The fourth level was the most challenging. The shapes were more difficult to construct using the turtle programme for description of their procedural part. The pupils had to use more complicated structures, i.e., nesting cycles, to fit their programme into the command line. After finishing each task level, some data usable to later analysis was saved so that the researcher could see the same situation on the screen as the pupil whenever the pupil asked the turtle to depict the programme. This enabled us to find common mistakes made by children as well as their strategies for tracing the given figure.

Course of the testing
The duration of the testing 90 minutes must not be exceeded. The test consisted of an introductory training (20 minutes) and four levels of the game-like activity. At each level, the tested pupil had to pass six tasks to fill a given figure. The main purpose of the training was to acquaint pupils with the environment of the Tracer application and to introduce them to turtle graphics and geometry through a simple way of using experiments and manipulations.

After a pre-test to confirm the testing application in school conditions, the organization of the test, and to verify that the data obtained within the testing would be sufficient, about 40 testers were prepared and the test was realized in 28 South Bohemian schools with 212 children. None of the tested pupils had any prior experience with Logo, and most had never used computers in school.

Outcomes of the testing
Programme creation and editing
Some pupils verify their programme very often, especially in the initial level. After each change, they verified the programme by letting the turtle draw an outline. Some of them evoked the feeling that they were randomly choosing any new command for the end of the programme, verifying it immediately, and then erasing it or leaving it in the programme. A typical example was turning the turtle in a concave angle of rectangle (Figure 6).

It appeared that pupils were not initially well-informed about their programs because they corrected longer parts more often than it was essentially necessary. Another strategy was to delete the programme and to begin creating it again from the beginning.

Strategy of polygon creation
While observing them we found that those pupils who seldom verified their programme by the drawing the outline helped themselves with the illustration of an imagined turtle motion with their finger on the monitor. These manners were less frequent at higher levels.
Observing the pupils in the main test resulted in the conclusion that pupils moved after approximately two experiments to the strategy of first paving a figure with tile objects (some pupils immediately from the first task, one group at the last one). In this way, they used the computer for drawing a rectangle as the remainder of a figure after paving with tile objects, and for the ensuing rest they wrote a programme to be drawn by the turtle.

Some children when unsuccessful tried the opposite direction of a turtle walk through a figure (clockwise/anticlockwise).

Overall, pupils have greater difficulties writing a programme than discovering the division of a figure into a procedural and a tile object part. Generally, children have no problems in the third level when they first have to finish the programme and then to pave the figure (No. 13–18 at the Table 1).

We could see many times that tile objects were placed correctly, but that the pupils were not able to write a programme to fill the remaining rectangle. It appears that a longer time period within which the pupils would handle the turtle and write a programme should transpire before allowing the pupils to solve tasks of the main test. This can be supported with a longer and more goal-directed training, and a longer first level of testing.

The statistical evaluation showed us several interesting shapes for analysis (Tables 1, 2 and Figure 5). A very small number of requests for executing the programme before checking at the task No. 12 doesn’t correspond with the poor success rate on this task. The shape used in this task might explain this discrepancy; it has no procedural part. Children were probably confused by this, giving up very quickly. – The task No. 15 is the first one with non-trivial procedural part at third level. When children don’t imagine the procedural part correctly, they aren’t allowed to correct it after checking (see Figure 5). A combination of these two used conditions makes this task difficult which corresponds with a low rate of correct answers. – Maximum of requests for executing is at the task No. 19, which is the first shape in the fourth level, with complicated shapes difficultly described by the turtle commands. – Decreasing number of children solving level No. 4 could be explained by lack of time.
Procedural and paving way of building-up of geometrical object in concepts and strategies...

Cycle and symmetry

As described by Tržilová, a participant of the research (Tržilová, 2007), only a very small number of pupils did not use the cycle at all when writing a programme. These others used the cycle only for repeating one command (Repeat 3x [Step]). Children who inserted more commands into a cycle (for example, … Repeat 2x [Step Left Step Right] …) could find a repeating part of a figure or the repeating sequences of programme.

Some pupils discovered the connection between the central symmetry and the programme which was all written as a cycle (for example Repeat 2x [Step Step Left Step Left]). These pupils tried to discover symmetry in further pictures, and to use a cycle as a general approach to writing a programme for such kind of picture.

Conclusion

Some findings for further investigation: how do children find a way to trace a given shape, and how do they develop an understanding of the turtle movement and its description? Several general strategies can be described.

- It is easier to fill in the shape with tiles than to describe it by a set of movement commands – it may be because the pupils have less experience with this type of expression.
- When children were not successful in tracing clockwise, they often started from the beginning in the opposite direction.

Pupils begin to use a cycle for notation of programme only when they are forced to shorten their programme.

Table 1. Task levels No. 1, 2. A number of answers for each task, a median of a number of requests for turtle drawing before checking and a percentual rate of correct answers.

<table>
<thead>
<tr>
<th>Task level</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>Task number</td>
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</tr>
<tr>
<td>Number of answers</td>
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<td>212</td>
</tr>
<tr>
<td>Median of request</td>
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<td>4,5</td>
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<tr>
<td>Rate of correct answers</td>
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<td>97</td>
</tr>
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</table>

Table 2. Task levels No. 3, 4. A number of answers, a median of a number of requests for turtle drawing and a percentual rate of correct answers.

<table>
<thead>
<tr>
<th>Task level</th>
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<th>4</th>
</tr>
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<td>14</td>
</tr>
<tr>
<td>Number of answers</td>
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<td>184</td>
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<tr>
<td>Median of request</td>
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<td>2</td>
</tr>
<tr>
<td>Rate of correct answers</td>
<td>85</td>
<td>91</td>
</tr>
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</table>

Cycle and symmetry

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- When children were not successful in tracing clockwise, they often started from the beginning in the opposite direction.

Pupils begin to use a cycle for notation of programme only when they are forced to shorten their programme.

Figure 6. A recorded process of a polygon creation. Maruška was cautious, she used the strategy “paving first” and checked her programme after any change.
Some pupils discovered the connection between central symmetry and the use of a cycle within the writing of a programme. There is a question whether children discovered the procedural part as an inscribed rectangular part of the shape. A new version of the *Tracer* application with a new set of tests has been prepared in which commands for only 60-degree-turns of the turtle are implemented. We hope that it allows us to manage an assessment of non-rectangular procedural parts of a shape and to decide whether children of this age can use not only right angle, whether a way to the concept of angle could go from right angle through familiarization with a concrete non-right angle.

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References

Promoting thinking through a modern approach
for problem solving

The paper introduces the participants to a unique way of problem solving involving algorithmic thinking in the mathematics classroom. Writing different algorithms to a given problem enhances a variety of thinking types one of which is algorithmic thinking and evokes several solving problem strategies. Writing the algorithms serves as a trigger for mathematical exploration, supports number sense ability and intensifies independent thinking of students.

Constructivists believe that the learner must be active, constructing his own knowledge, integrating new information into his schema thus promoting new thinking abilities. Thus, not only ready-made software is to be used for solving mathematical problems but, that the algorithmic computational programs leading to the solution should be prepared by the students (Breuer & Zwas, 1993). The students write Excel programs for their implementations and then run them on the computer.

I. Polya (1957) describes four stages for problem solving:

- understanding the problem
- devising a plan
- carrying out a plan
- looking back

We adapted this model throughout our course for pre-service teachers in the following way enhancing different thinking types as seen below:

<table>
<thead>
<tr>
<th>problem</th>
<th>intuitive thinking</th>
<th>analytic thinking</th>
<th>independent thinking</th>
<th>algorithmic thinking</th>
<th>heuristic thinking</th>
<th>recursive thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>algorithm</td>
<td>intuitive thinking</td>
<td>critical thinking</td>
<td>reflective thinking</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>computer program</td>
<td>solution analysis</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is possible to show how this model is carried out throughout a technology based course which deals with solving algorithmic problems using computers. We will discuss the various types of thinking mentioned above in connection with the four stages of the model. We can focus on problems for pre-service teachers for the elementary school. (For pre service teachers who are intended to teach at the middle school or high school we developed two other courses making suitable adjustments).

This course was developed as a result of our efforts to integrate computers into mathematics classes in teacher training colleges, and as a result, to promote teaching algorithmic problem
solving as early as elementary school when they become practicing teachers. Former studies (Hoffmann & Klein, 2007) showed that the subject can be adjusted for elementary school pupils. Most pupils were able to write an adequate algorithm to a given problem and to ‘run’ it on a computer using an Excel spreadsheet.

We elaborate on the four parts of our adaptation to Polya’s model:

**The problems and the algorithms**

The problems were chosen either from the materials taught in the elementary-school (pre algebra stage) or higher level problems targeted for the pre-service teachers. The students learn to write (and read) various kinds of algorithms for infinite sequences followed by algorithms for finite sequences taken from Discrete Algorithmic Mathematics, such as: The triangular numbers, the square numbers, factorial numbers, Fibonacci numbers, the golden section, division with remainder, Ancient Egyptian multiplication, or Euclid’s algorithm for finding the Least Common Denominator of two given numbers.

In the advanced courses other problems are introduced where students need to develop numerical methods to deal with new and vital subjects, ordinarily absent from the regular school programs, such as: approximate solutions and calculating to a desired accuracy, computing the square and cubic roots for a positive number, or various numerical methods for computing the digits of Pi.

**The computer program and the solution**

After writing the algorithms students translate them to computer language using Excel spreadsheets. While using computers student expand their knowledge in Computer Science and are exposed to computational errors which lead to mathematical contradictions. They deal with the influence of a small deviation of the input on the output, rounding errors, rate of convergence etc.

**References**

Preparing of pupils for notion of limits

Brief texts on independent thinking in mathematical education
From a caterpillar to a butterfly:
a learning project for kindergarten

Geometry takes origin from motor, visual and tactile experience (Enriques, 1901). The geometrical concepts are generated from spatial investigation, observation and manipulation of tangible objects. Using photos and other materials, the poster presents a learning project, prepared with the aim to introduce kindergarten’s children to geometrical concepts and not only. At the beginning, a simple tale, named “The tale of Pelù-pillar” about a furry caterpillar is employed. The tale is wrote in a particular book¹, “illustrated” with pipe-cleaners glued on the pages. In this way the reader can distinguish by touch different shapes, that correspond to different positions, described on the book, assumed from the caterpillar: segment lines, curves, circles and so on. After the reading, the teacher propose a linguistic activity: use appropriated expressions with the aim to identify the different positions of the grub. Afterwards pupils are asked for reproduce with their body the positions of the caterpillar. This implies a dramatisation activity and a planning of behavioural patterns. From the mathematical point of view, the transformations of Pelù-pillar (concerning the position in the page or its length or its shape) suggest the idea of geometrical transformation, an important concept of modern geometry. Children can explore the concepts of deformation and closed or open line (topology), of rotation (isometry), of “horizontal” and “vertical” line, of congruence, and so on. Moreover the positions like “C” or “U” suggest a link with alphabetical letters. The path continues with an iconic activity: the pupils draw the different shapes of caterpillar, copying them from the book. In this way each child have a pile of papers for play. The teacher propose various kind of game: ‘found the right paper’, composition of sequences, game of ‘straight’ or ‘curved’, game about left or right, on or under.

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¹The book is conceived from E. Alberici (Association of Social Promotion “Contatto” di Traversetolo (PR–Italy).
Children’s models of number 3 at the age of 6–7

This paper is result of one experiment which we focused on the 6 and 7 years old children attending the first class at the elementary school. We explored the level of their number conceptions and the ability to use them. For the realization of the experiment we chose number 3, because it is the number from the numerical scope 1–10 and as it is freely distributed in fairy tales, advertisements and other areas of ordinary life.

The experiment was realized in January 2008 with the group of 17 children in the 1.B class at the primary school in Nábrežie Mládeže Street in Nitra. Children were given blank papers and were asked to draw what they imagine, when we say number three. We pursued the ability of children to assign the specific model to a given number. All the pictures were fixed to the board. Children were asked to arrange the pictures into groups according to some similarity and to explain their choice. Thus the concept maps of children concerning their images of number three were created. Children created the fairytale with number 3 from their models.

Ternary Fairy Tale

Princess Cindy III. (the Third) lives on the third floor of her castle. She likes to wear her golden triangular crown on her head, and most of all she likes to drive her car with number 3. One day, she took a three-coin from the piggy bank and went to buy her favorite chocolate bar 3BIT. On the way back, she had to stop at the traffic lights and there she met her friends – three pigs. They arranged to build a snowman together. They took a three-wheel barrow and carried three snowballs of the snowman in it. Behind the three mountains, there lived one bad triple-headed dragon that suddenly flew in and kidnapped Cindy III. Frightened three pigs ran for the help to three Spidermen. They did not hesitate, took their three magic stones and flew to save Princess Cindy III. They slayed each of the three dragon’s heads with one of the stones, and flew away with happy Princess Cindy III. There were already the three pigs waiting for them. After that, they played “PoGs” all together. Te winner obtained three golden “PoGs”. On the third tower there hung the three bells, and the fairy tale came to its end.
On the numerical competence of six-year-old children

The investigations covered by the poster presentation focus on the numerical competence of six-year-old children, one year prior to their undertaking compulsory education at school. The aim of the research was finding answers to the following questions:

– What do the children who have not begun systematic education at school yet associate the word ‘count’ with?
– To what extent they have mastered the sequence of initial natural numbers?
– If and how they use this skill to determine potency of a set, to create a set equipotent with another one, to compare potency of two sets, and to divide a set into two equal subsets.
– What difficulties do the children face when using counting competence skills?

The research was conducted with the method of individual interviewing in December 2007. It was a repetition of the research conducted in September 1983. It allowed for comparing numerical competence of six-year-old children in the two periods.

The results of the research conducted in the two rounds (1983 and 2007) are similar and they show quite a wide, although rather differentiated, numerical competence of the investigated children:

– When answering the question “Can you count?” all the children reel off successive numerals at least up to ten, almost half of them up to one hundred, and some of them have consciousness that “this counting will never end”;
– A majority of the children can correctly use a known sequence of numerals to count particular things;
– Normally, children are critical and correct in evaluating their own arithmetic skills, and often (apart or instead of the procedure of counting, they apply non-numerical procedures.

Numerical competence of six-year-old children is differentiated: in many of them it includes some knowledge and skills comprised in the syllabus of the first class (and sometimes higher), but also in many, in certain situations, it reveals immaturity for learning mathematics in a “school manner”.

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On mathematical education of 6 year old children in Ukraine

Preliminary courses (children of age before 6) in Ukraine are devoted mostly to the notions “more-less”, “higher-lower”, “left-right”, counting up to 10, solving simple problems concerning addition and subtraction up to 10 like: “3 apples + 2 apples = 5 apples”, “3 apples – 2 apples = 1 apple”. Words “good”, ‘very good” etc are used instead of marks.

There is no special program for this period. These courses are not obligatory in Ukraine and there is no data what number of children enter the courses (this number depends on regions).

The first form (class, 6 year old children): there are two basic subjects, namely, mathematics and language (reading).

The program of mathematics for the first form pupils includes:
- numbers from 1 to 10;
- properties of objects;
- geometric figures;
- quantities;
- addition and subtraction of numbers from 1 to 10;

At the end of the first form, the pupils should count to 20, add and subtract up to 10, know the terms for the components of the addition and subtraction, geometric figures (point, curve, line), polygon, triangle, rectangle etc., the length, mass and volume units, their notations.

There are two basic textbooks for mathematics in the first form.

Because of importance of mathematical education, the number of teaching hours for mathematics increased 1 hour per week during the academic year 2006/07. This allowed for distributing the teaching time in a more rational way. In particular, the topic at the beginning of the first form consists of the material which has been already studied. This will ensure more effective understanding of the material.

The increasing of number of teaching hours allows for wider using game and practical teaching methods, which, in turn, will simplify the process of adaptation of pupils to school. The teacher will have a possibility to pay more attention to the level of mathematical knowledge of pupils and ensure the individualization of the process of studying. The teaching time can therefore be varied in accordance to the level of pupils.

The lesson of mathematics in the first form is usually combined and consists of several parts. Each of these parts has its own logic of studying and methodology of teaching. Such a structure of lessons is relatively new and is a subject of pedagogical investigations. A small number of parts allows for avoiding chaos in problem solving and for deep considerations of program tasks. In addition, thematic lessons are now more and more often used in pedagogical practices.

One of the main aims in studying mathematics by 6 year old children is forming of knowledge how to solve mathematical problems. The twofold essence of problems consists in the fact that they are a part of the program in mathematics and, on the other side, that they are a didactic tool for studying of mathematics.
Percentages are not another name for fractions

One of the commonly used mathematical subjects in everyday life, and in sciences, is percentage. Therefore, it is our duty, as educators, to take care that our students will possess a well-founded understanding of this subject. Unfortunately, we found that not the case (Sheffet & Bassan, 2004).

There are two main misconceptions held by students.

The first is: Percentages are another name for fractions.

Percent is one-hundredth. Percentages describe part of a quantity and are not numbers such as fractions. Fractions have many functions, only one of which is describing part of a quantity. A percentage can be replaced by a fraction and vice versa, but only when the fraction describes part of a quantity.

When a person believes that fractions and percentages are the same, he may use percentages in the same manner he uses fractions. For example: It is true that if one number is bigger than the other by 1/2 than the other one is smaller than the first one by the same 1/2. Following this knowledge, this person may believe that if one number is bigger than the other by 50% than the other one is smaller than the first one by the same 50%. But the last one is false.

The second misconception is: In percentage problems, 100 is always the denominator.

Elementary percentage problems are of three principal types: 1. Calculating the value of the percentage; 2. Calculating the percentage; 3. Calculating the fundamental size (van-Dijk et al. 2003). Only at first type problems 100 is the denominator. At the other two types 100 is the nominator. Still, students use to hold the idea that 100 is the denominator. Whenever they meet an equation with 100 as nominator they believe it is a false answer.

References


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The method of incomplete pictures and pupils’ ideas

Language allows us to see the cognitive process of children (Gray, Pitta, 1996).

Method of incomplete pictures is based on this idea
The pupils (age 10–11) were tested separately without a time limit. Instruction:
Write what you see in the pictures. If you need to finish a picture, you can do that.

The pupils see each of the incomplete pictures as properties of objects: real and imaginative (the outside world and experiences of a child), mathematical – geometric and non-geometric, symbolical.
The answers show – first, the attitude of children to mathematics and its teaching process – secondly, the ways how to motivate them to learn mathematics. Real or imaginary objects prevail with some pupils (see sample B) in the other cases mathematical objects are more frequent (A). The link between mathematical images on one side, and the real world and the children’s inner perception of the world on the other side, make for the forming of a correct mathematical concept and its true position in the cognitive structure. Resulting from a carefully chosen set of incomplete pictures this method can be used to diagnose a pupils’ idea of a particular concept. This is the reason why the method deserves further development.

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2. With the support of GAČR 406-08-0710.
From research on understanding a letter symbol by students of a junior high school

The research deals with understanding a letter symbol by students of the second grade of a junior high school. Its aim was to examine how a sample group of 25 students understands certain issues related to letter symbolism and what difficulties they have in this respect. The goal was also to determine hypothetical causes of observed difficulties.

As a research tool a set of adequately modified tasks which come from a teaching proposal of a functional approach to a letter symbol (Kusion, 1993) was applied. The tasks regarded the linear function and, in the context of this subject matter, they were intended to provide the possibility of recognizing the understanding of the following issues related to letter symbolism by the students:

- Different letters can denote the same number;
- Symbol $x$ does not have to necessarily denote a number from a set of real numbers;
- The formula $y = \ldots$ does not always have to applied to describe a constant function; depending on the denotations used, $t = 5$ can be an example of a correct notation;
- A variable can be substituted by not only a specific number, but also any other variable.

An analysis of the solutions provided by the students, followed by individual interviews with the subjects, suggests certain conclusions (Michalska, 2007). Among others, it indicates student’s attachment to the letter symbolism as used during their classes, a mechanical use of a symbol and difficulties in understanding the conventionality of symbols.

References

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Language structures of pupils  
within problem posing and problem solving

Language structures are understood as a complex of semiotic systems of representation and rules which guide their construction, interpretation and application. Existing results of investigations of pupils’ representations and their transformations in the teaching of mathematics motivate us to look into semiotic systems of representation and rules which guide their construction, interpretation and application, not individually but in a complex way as a language structure. At the same time it is necessary to consider the context which the language structures are grounded in. The quality of language structures is the key factor of mathematical literacy. On the basis of representatives and relationships between them is one of the ways to diagnose pupils’ understanding. The pupils’ representatives characterise their semiotic systems of representation of mathematical objects. At the same time, they provide some information about the mental representation (images) of these objects.

A typical aspect of the teaching of mathematics is problem solving. Researchers in mathematics education, however, also study problem posing which together with their solution represents an important diagnostic tool (Silver and Cai, 1996). Based on the problem which was created by the pupil, we can infer evidence about his/her understanding of the mathematical concepts and relations in question. Problem posing is connected to the transformation of the given problem into different systems of representation (Leung, 1997), therefore, it is a suitable didactic situation for investigating language structures.

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Some geometric problems which help to overcome a pupil's tendency to use formulas

The paper introduces such tasks that would help pupils apply geometric skills rather than algebraic ones. In particular, the tasks are constructed to be easily solved by using a geometric view. Applying algebra can lead to a more sophisticated level of mathematics or may not result in a solution at all. The majority of both pupils and teachers is more prone to use algebraic tools that can be learnt by heart or looked up in the tables. That is why we feel that it would be beneficial to point out such problems that would develop teachers’ and pupils’ ability to see in geometry. Three problems will be presented.

The first task is called “Garden Fence”. Pupils are asked to “straighten” a bent line that represents a fence between two gardens without changing their areas. Since no parameters are provided, it is not possible to apply any simple algebraic procedure. Pupils are encouraged to show their knowledge of triangles and resolve the problem using a geometric insight.

The problem number two, “Path Across a Field”, depicts a rectangular field across which pupils are to design a pathway. Two possible paths are available and pupils need to choose the one with a smaller area. Again, the task cannot be solved by algebra because all the data but the width of the path are unknown. The task is based on the knowledge of the congruence of triangles. In the end, pupils should be able to conclude that both pathways are identical in their areas.

The last problem deals with a relationship between line segments within a circle. Even though the radius of a circle is given, the algebraic solution is once again not available. On the other hand, a geometric answer is very clear once a pupil determines a rectangle and its diagonals in the circle.

Similar problem solving activities should help overcome a pupil’s tendency to use algebra and apply geometry more often. In addition, geometry should be perceived as a tool for solving problems and not only as a straightforward drawing using a ruler and a compass.
Development of functional thinking

Functional thinking is an important component of the teaching of mathematics at primary and secondary schools. The paper shows a series of tasks which focus on the development of functional thinking of lower secondary pupils and illustrate their work by examples. The tasks were developed within a Socrates-Comenius project IIATM and tested in the third grade of Prague Eight-Year Secondary Grammar School in standard lessons of mathematics with 31–33 pupils (13–14 years old).

The series of tasks is as follows:

1. *Growth curves of population*. The pupils solved the task: What do the graphs in the picture mean? (Describe them.) They were given three graphs taken from biology without any indication of what the axes might mean.

   The pupils discussed a wide range of interpretations which enabled the teacher to diagnose what their current understanding of graphs of functions is.

2. *Graphs of real events*. The second task was assigned as homework: Find examples of graphs which describe some real events or situations.

   The pupils found different kinds of graphs and diagrams in newspapers and on the internet.

3. *Properties of functions*. This stage consisted of two tasks: 1) Brainstorming – What properties can a function have? 2) Investigate the given graphs and diagrams from the point of view of their properties.

   The goal of the first part was to find out what concepts concerning functions the pupils have and whether they know their proper mathematical names. In the second task, the pupils investigated some graphs and diagrams (which the teacher chose from among those which they brought earlier) from the point of view of their mathematical properties.

4. *“What does the graph say?”* The task was: Describe the relationship in the given picture.

   This task was used again as a diagnostic one and the pupils were to use their acquired knowledge and show what they had learned.

   During the above process, the pupils started from their real life knowledge which they were to apply in mathematics. Then, the mathematical properties were discussed and correct terms introduced. Finally, the pupils reasoned within mathematics only and applied the knowledge on a concrete graph again. The teacher could see the differences between their initial interpretation in the first task and their interpretation in the final task.
Developing the teaching of mathematics: Looking at continuing professional development (CPD) initiatives

The data that form the basis of this paper are being gathered as part of the Researching Effective Continuing Professional Development in Mathematics Education Project (RECME). The overarching aim of RECME is to explore the interrelated factors contributing to effective professional development, where we see CPD as a ‘patchwork of opportunities – formal and informal, mandatory and voluntary, serendipitous and planned’ (Ball & Cohen 1996). The intention is that the findings of the RECME project will inform the recommendations for future initiatives of NCETM and so help to develop opportunities for teachers to increase their expertise in the teaching of mathematics at all levels.

Over thirty ongoing CPD initiatives, which include more than 250 teachers, form our sample. The CPD initiatives that form the basis of this paper are a subset of this group and have been selected on the basis that they show contrasting approaches to working with teachers of pupils in the 5 to 14 age group.

However they are all engaged in working with teachers over an extended period of time and they are all running this academic year. They have been developed by different groups of people for different purposes and use a variety of different models. The purpose of the paper is to consider how these different models work and how the teachers taking part in these initiatives respond to the purposes of the initiatives. In particular the paper will address the ways in which the different initiatives work with teachers to develop the teachers’ ways of working with children that help the children to engage more deeply with mathematics.

Data for the case studies will be collected in a number of ways from the leaders of the professional development initiatives and the teachers involved. These will include the observation of sessions in which leaders and teachers are engaged; the collection of data about the projects through online questionnaires; interviews with the teachers and leaders and observations of some of the teachers’ mathematics lessons in their own classrooms.

References

How do practising teachers use projects in their teaching of mathematics?

One aspect of the on-going reform of teaching in the Czech Republic concerns the change of teaching approaches (not only in mathematics). Among others, practising teachers are encouraged to use *projects* in their mathematical lessons to make them more lively and enjoyable for pupils and to connect pupils’ learning of mathematics with their life experience. However, when we look into literature, we can see that the term project is understood differently by different authors. The question is how practising teachers without any formal introduction into teaching with projects cope with them in their teaching.

Last year, we organised a series of courses (within ESF programme) a part of which was a module on project teaching prepared by Marie Kubinova and Miroslav Hricz. The participants, elementary and lower secondary teachers, underwent 6 lessons of teaching and then were to write a report to get an official certificate. The majority of reports concern a suggestion of a project together with some practical experience from using it in the classroom. We have used this unique opportunity to investigate how project teaching is understood and used not by educationalists but by practitioners.

Namely, we have analysed 85 written reports of teachers to answer the following questions:

1. What is the teacher’s primary motivation for using a project?
2. What do practising teachers consider to be a project? Do they refer to any authority or do they understand it intuitively?
3. Can a project be a «purely» mathematical one or is it always interdisciplinary? What school subjects are mostly used? What kind of pupils’ life experience is mostly required?
4. In what ways are projects actually organised in the class?
5. Are the projects used to introduce new mathematical knowledge or to revise it? What mathematical knowledge is «prone» to be used in a project?
6. How do the teachers solve the problem of evaluating pupils’ work? Do they evaluate the use of projects as for their impact on pupils’ mathematical learning?

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