Teachers' professional development
I start this paper by considering different conceptions of mathematics and showing how exploring and investigating are central to the mathematical activity. Next, I present several examples of students investigating mathematics in the classroom that illustrate important aspects of an exploratory approach to mathematics teaching and learning. Such an approach does not depend only on the nature of tasks, but requires an analysis of the roles of teachers and students, the communication patterns in the classroom as well as the overall organization of content and processes in meaningful mathematics teaching units. I indicate that this kind of teaching is rather demanding and refer three main conditions to help teachers in developing professionally to carry it out: collaborating, researching their own practice and getting involved in the professional community. I close with a brief discussion on the relationships of investigating, teaching, and learning. I argue that as students explore and investigate mathematics, teachers need to investigate their own practice in professional collaborative settings.

1. Conceptions of mathematics

There are many perspectives about mathematics. Most dictionaries present it as the science of number and form (Davis & Hersh, 1980). For many mathematicians, mathematics is the science of proof. We may recall the famous saying of Bertrand Russell: “mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true” (Kline, 1974, p. 462). Jean Dieudonné put the same idea in a shorter way: “qui dit mathématiques, dit démonstration”. The structuralist movement of the first half of the twentieth century encouraged the view of mathematics as the science of structures, and that framed the program of Bourbaki and influenced a deep educational reform in the 1960s known as “modern mathematics”. Still another view claims that mathematics is best described as the science of patterns, aiming to describe, classify and explain patterns in manifestations such as number, data, forms, organizations, and relations (Steen, 1990).

When we think about mathematics we may focus on the body of knowledge embodied in articles and books or in the activity of people doing mathematics. Regarded as an activity, mathematics is indeed a dynamic science. That is well captured by George Pólya (1945), who says “mathematics has two faces; it is the rigorous science of Euclid, but it also something else [...] Mathematics in the making appears as an experimental, inductive science” (p. vii). It is also sustained by Imre Lakatos (1978) that states “mathematics... does not develop through monotonous
Mathematics can be an interesting activity not only for the mathematician but also for the teacher and the student. Singh (1998) refers that Andrew Wiles, now famous for his proof of a long standing theorem, recalls the role of his teacher in getting him involved in mathematical explorations:

Since I found for the first time Fermat’s Last Theorem, when I was a child, this has been a major passion... I had a high school teacher who did research in mathematics and gave me a book on number theory and provided some hints on how to attack it. To begin with, I started from the hypothesis that Fermat did not know much more mathematics than me... (p. 93)

Another mathematician, Jacques Hadamard (1945) states that there is no major difference in the mathematical activity of a student and a mathematician when they are solving problems and exploring mathematical relationships:

Between the work of the pupil who tries to solve geometry or algebra problem and creative work [of a mathematician], we can say that there is a difference in degree, a difference in level, both having a similar nature (p. 104).

In mathematics, the starting point for an investigation may be a mathematical or a non-mathematical situation from other sciences, technology, social organizations, or daily life. As an activity, a mathematical investigation includes the formulation of questions, the production, testing and refinement of conjectures, proving and communicating results. Carrying out a mathematical investigation involves conscious and unconscious processes, aesthetic sensibility, and connections and analogies with mathematical and non-mathematical situations. It is undertaken by different people with different cognitive styles – analytic, visual, conceptual (Burton, 2001; Davis & Hersh, 1980).

2. Students investigating mathematics in the classroom

Let us consider some examples of students working as mathematics researchers.

Example 1 – Working with numbers. The first example comes from a class led by Irene Segurado, a grade 5 teacher working with 10 year old students (see Ponte, Oliveira, Cunha & Segurado, 1998). The task is the following:

1. Write in column the 20 first multiples of 5.
2. Look at the digits of the units and tens. Do you find any patterns?
3. Now investigate what happens with the multiples of 4 and 6.
4. Investigate with other multiples.

This task was presented at the beginning of a 50-minute class. The teacher had planed group work, but she found the students very agitated at the beginning of the class and decided to redefine her strategy and work as a whole class. A list of multiples of 5 was written on the board and students begun looking for patterns:

Tatiana, raising her arm, answered quickly: The units’ digit is always 0 or 5, and that was accepted by her colleagues, echoing around the room: it is always 0, 5, 0, 5...
Teacher: What else?
Octávio, with a happy face: The tens digit repeats itself: 0–0, 1–1, 2–2, 3–3...
Carlos, agitated: I discovered something else... May I explain at the blackboard? (…)
As he got to the blackboard, he explained: 0 with 5 is 5, 0 with 0 is 0, 1 with 5 is 6, 1 with 0 is 1, 2 with 5 is 7, 2 with 0 is 2, 3 with 5 is 8, are you getting it? There’s a sequence. It’s 5, it jumps one, it’s 6, jumps one, it’s 7... Or it’s 0, jumps one, it’s 1, jumps one, it’s 2... (Ponte et al., 1998, pp. 68–69)

We see that the students were able to identify different kinds of patterns. They noticed simple repetition patterns (such as 0 5 0 5…) and more complex patterns combining linear growth and repetition (such as 1 1 2 2 3 3…). They also combined different elements to identify linear patterns as subsequences of rather complex patterns (0 5 1 6 2 7 3 8 …).

The class also analyzed patterns in the multiples of 4. Then, they turned to the multiples of 6 that were put in a column side by side with the multiples of 5 and 4.

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Students’ discoveries were coming in bunches. They were rather excited, thus creating some difficulties to the teacher in recording and systematizing their contributions:

- The units’ digit is always 0, 6, 2, 8 and 4.
- The units’ digit is always even.
- The tens’ digit does not repeat from 5 in 5.

The teacher tried to handle this enthusiasm: *Take it easy! Let us verify if what your colleague said is true. Attention! Look! Look how interesting what your colleague discovered!* Suddenly, Sónia said: *There are the same digits that for the multiples of 4.* Even before this statement made any sense to the teacher, Vânia continued: *But they are in a different order.* The teacher figured out that the students were comparing the multiples of 4 and 6, and she explained that to the class. Other students went on:

- It also begins with 0.
- The other digits are in a different order.
- There are multiples of 4 that are also multiples of 6.
- The multiples of 6, beginning at 12, are alternately also multiples of 4.

The students could find again complex repetition patterns (such as 8 2 6 0 4 8 2 6 0 4 8…) and, more interesting, they were able to compare features of different patterns. In this activity they developed their number sense, they got a better grasp of the behaviour of multiples, and they did a lot of mental computation. They expressed their generalizations in natural language.
In her reflection, Irene indicates that the students surpassed all her expectations. She says: “I had not foreseen the hypothesis of comparing the multiples of the different numbers, because I had never put them side by side. Therefore, I experienced their discoveries with great enthusiasm” (p. 71). She also reflects on the implications of working as a class, as compared to small groups: “The contribution of a student was ‘picked’ by all his colleagues, yielding a greater number of discoveries” (p. 72). It would seem that in curriculum topics such as multiplication facts, multiples, and divisors, at the elementary school level one can just do routine exercises. This experience shows that, on the contrary, these topics allow for much exploratory and investigative work.

**Example 2 – How is the typical student in my class?** A second example comes from a class of Olívia Sousa, a grade 6 teacher working with students aged 11 (see Sousa, 2002). The task was organized as a statistical investigation: “Imagine you want to communicate to another student in a distant country, or, who knows, to an ET, how students in your class are?…” This was meant to have students taking all kinds of measurements about their bodies and collecting data about their families, which usually raises high levels of students’ enthusiasm.

Five 90-minute blocks were planned to carry out this task, with students working in small groups. In practice, six blocks were necessary. The teacher divided the whole task in four main steps: (i) preparation of the investigation questions; (ii) data collection; (iii) data analysis; and (iv) reporting the results. In each step some written instructions were provided to the students. See below, for example, the directions for step 2.

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With your colleagues:
• Write as a question each one of the characteristics that you are going to investigate.
• What answers do you expect to obtain for your questions?
• How (through observing, measuring or a questionnaire) can you get the answers to your questions?
• Prepare data sheets to collect the data.
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In this class the statistics measures (mean, median, mode) had not been taught before. A major decision in this experiment was to have the students working with their previous knowledge of these measures, instead of teaching them first and after propose that they do application exercises to practice. Therefore, along with several other directions, the students were asked to find the mode (indicated as “the most frequent value”), the median (the “middle” value), and the mean. They had no trouble in finding the most frequent value. To find the median took more time, but when they realized that they could order the values, it became a rather easy task for most students. There were a few problems as some students forgot to count repeated values or took the median as the average of the extremes. The class discussion was a good setting to sort these things out. The students had already a strong intuitive notion of mean as the result of summing two values and dividing by 2:

Inês: Then we put 1 and 35.
Alexandre: 1 and 40.
Prof. How did you get 1 and 35?
Inês: Pardon?
Prof. How did you made that 1 and 35?
Inês and Estelle: It was an estimation!
Inês: It is not as Mauro (1,20 m) nor as myself (1,50 m). It is in the middle.
Estelle: It is between.
Inês: It is between the two.
Estelle: Mauro and Inês.
With the help of the teacher, they were able to generalize this intuitive notion to find the mean of more than just two numbers.

In her reflection, Olivia considered that carrying out this task was a significant learning experience, of an experimental nature, in which the students worked mathematics contents of two domains, statistics and numbers and computation, in an integrated way. Decimal numbers, obtained from measuring quantities associated to the body, were no longer abstract quantities and acquired a strong meaning. The manipulation of these numbers in a significant context—comparing, ordering, sorting, and operating—contributed towards students improving their global understanding of numbers. Olivia considered that, regarding statistics topics, the contact with different kinds of variables and with different ways of collecting, organizing, and representing relevant and meaningful information, promoted students’ understanding of the statistics language, concepts and methods that went much beyond simple memorization. This example shows that undertaking an investigation based on the students’ reality can be the starting point to develop investigation competences, to learn new mathematics concepts (in this case, statistics concepts), and to practice and consolidate existing mathematics knowledge.

Example 3 – How to amplify? The next example concerns an experience carried out by João Almiro (2005) a grade 8 teacher. The task is indicated in figure 1.

The Visual Education teacher wants to amplify the picture below but she puts the following condition: the area of the amplified picture must be 400 times larger than this. The teacher is going to do a overhead transparency with the picture and project it in the wall. But she has a big problem: At what distance she must put the overhead projector from the wall? How can we help her? Write a report that includes the description of your investigations, the computations that you made, your conjectures and possible solutions.

![Figure 1. How to make it 400 bigger?](M.C. Escher, 1965)

The students had to design their own strategies. The teacher prepared the room with four overhead projectors (each one to be used by two student groups) and gave a metric strip and a ruler to each group. The room was a little small for the projectors but work was possible anyway. The teacher did not provide further instructions, just said that it was a request from the Visual Education teacher. Of course, many students did not believe this.

The reactions from the groups were very different. Some were lost, not knowing what to do. As one student wrote in a final questionnaire: “I felt some difficulties with the overhead projectors since in the beginning we did not know where to start”. Others, immediately started trying to find ways of doing the task. The teacher was pleased to notice that all the groups understood that the projected rectangle would need to have length and width 20 times larger that the initial picture, so that the area was 400 times larger. The students had solved problems involving enlargements before and were able to mobilize this previous knowledge.
The big difficulty of the students was finding the distance that they should put the overhead projector from the wall so that the length and the width amplify 20 times. Almost all the groups constructed a rectangle with the dimensions of the picture. They projected, they measured what they found, and then they figured out how many times the length and width were now larger. They quickly understood that they did not have space in the room to enlarge the dimensions 20 times and, therefore, they had to use a computational strategy to know what distance the overhead projector had to be from the wall.

In one of the groups, the students understood that there was a direct proportion between the distance of the overhead projector from the wall and the number of times that the dimensions were amplified and quickly solved the problem. Other four groups, however, had much more difficulty. Helping each other, they went on measuring and arguing and when a group arrived to a conclusion it was shared with the others. They progressed sometimes making conjectures that the other groups refuted and proved that were not correct. They arrived to solutions that the teacher considered acceptable. This is the final part of the solution of one of the groups that used the notion of unit rate and the cross product:

![Figure 2. Unit rate and cross product to solve the amplification problem](image)

Measuring the picture, they found that it was a rectangle with 11.2 cm by 7.9 cm. Enlarging the length by 20 yields 224 cm. As they found that with the projector 1 m from the wall transformed this length in a segment with 44.5 cm, using the cross product, they found the required distance. For three other groups this was a very difficult task, and they were not able to do it, even with frequent help from the teacher.

Some students (about 1/5) reported a negative view of these classes. One of them wrote: “I didn’t like these classes (…) I think that I learn more in classes doing exercises and asking questions”. However, other students were pleased and recognized that they had significant learning. As one of them said:

The problems are a bit more complicated that those from other classes, at lest the overhead one, in which we had to think a lot, develop, we had to think different methods, to achieve the ideal method to get the correct result. We had to begin by finding out what was to do. In textbooks, the questions are direct, they tell us immediately what we have to do.

These responses from students show that not all of them will get very excited when the teacher presents challenging tasks. It is not because of “motivation” that these tasks have an important role in mathematics teaching. It is because they may promote significant learning. Working on this problem the students were called to draw on their previous knowledge of similarity, area, and direct proportion. They also had to design a strategy to collect data to figure out the relationship of the distance of the overhead projector to the wall and the size of the image.
Example 4 – Numerical equations. This example is drawn from a teaching experiment carried out by Ana Matos (2007) in her grade 8 class, involving an algebra teaching unit. This teaching unit included the study of numerical sequences, functions, and 1st degree equations. The class included a high number of students that were recent immigrants from countries such as Angola, Brazil, Cap Verde, Guinea, S. Tomé and Prince, and Romania.

The unit was carried out in 16 classes (90 minutes each). It included several kinds of learning experiences. The first part of the unit included exploratory and investigative tasks (Ponte, Brocardo, & Oliveira, 2003), as a mean to foster the construction of new concepts. In the tasks about numerical sequences, the students had to explore numerical patterns, with different levels of difficulty (some of which presented pictorially). These tasks created opportunities for identifying generalizations, which could be expressed in natural language at first but should progressively be expressed using algebraic language. In this part of the unit, letters were mainly used as generalized numbers and as unknowns in simple 1st degree equations. This is the overall plan:

<table>
<thead>
<tr>
<th>Blocks (90)</th>
<th>Topics</th>
<th>Objectives</th>
<th>Aspects to develop</th>
<th>Tasks</th>
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</thead>
<tbody>
<tr>
<td>3,5</td>
<td>Numbers yet – Number sequences.</td>
<td>• To discover relationships among numbers; • To continue sequences of numbers: divisors;</td>
<td>• Searching patterns and establishing generalizations; • Representing numerical relationships in natural language, by other means and symbols;</td>
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<td>3</td>
<td>Functions – Tables; – Graphics; – Functions defined by an algebraic expression. Direct proportion as a function. Graphics of the functions and.</td>
<td>• Read, interpret and construct tables and graphics for functions such as, or other simple ones; • Relate in intuitive way the slope of a line with the rate in a function such as ( y = mx ).</td>
<td>• Constructing tables of values, graphics and verbal rules representing functional relationships; • Understanding the use of mathematical models of real world situations.</td>
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<tr>
<td>6</td>
<td>1st degree equations – Equations with denominators and parenthesis;</td>
<td>• Interpret the statement of a problem; • Translate a problem by an equation;</td>
<td>• To particularize relationships among variables and formulae and solving simple equations; • To solve problems represented by equations and to carry out simple algebraic procedures;</td>
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Textbook exercises and problems

Textbook exercises and problems
In the second part of the unit, the study of functions was introduced by two tasks involving relationships between variables. Although the letter is used both as a generalized number and as an unknown, in this part of the unit the focus was on its use as a variable and on the notion of joint variation. In the third part, tasks 7 and 8 continued the study of equations that the students begun at grade 7 and revisited in previous topics, solving new kinds of problems and equations with denominators. In this phase, letters were mostly used as unknowns and as generalized numbers. All of the tasks allowed the students to use different strategies and to draw their own paths of exploration. This approach stimulates students’ active participation, providing them multiple entry points, adequate to their ability levels.

Working with sequences and functions became an opportunity to use the algebraic language as a tool for generalizing and sharing meanings. The study of these topics generated the opportunity to solve simple equations, which was important to create a common understanding among students, allowing them to continue learning more complex algebraic ideas. In the first general discussion, the sequence with general term $3n + 5$ was considered. The following dialogue took place:

Teacher: So, which was the order in which 300 was placed?
Erica: Teacher, $3 \times 100$...
Teacher: OK, but does that give 300?
Erica: No, that is just with $3n$.
Teacher: Oh, but I can’t change the rule like that because we would be working with another sequence, different from this one. We just need to know which is the $n$ that makes this expression yield 300.
Sofia: 300 – 5? I don’t know. [Students talk with each other.]
Erica: So, we make $3n = 300 – 5$.

Some students did not follow the reasoning proposed by Erica, and went on thinking on their own strategies. Pedro claimed with enthusiasm: “$3 \times 98 + 5 = 299; 3 \times 99 + 5 = 302$. It will not pass on 300!” This discussion continued with the contributions of Isabel, who solved the equation at the board, using her previous knowledge. The discussion provided a contrast between Erica’s idea, the formal solution proposed by Isabel and the intuitive process used by Pedro to see if 300 was a term of the sequence and the advantages of each of the processes.

This example shows how students may be encouraged to design their own strategies and how these may be discussed and contrasted in the classroom. Such discussion helps students...
to realize more connections and relationships and to become more resourceful to deal with new problems in the future.

An important feature of this teaching unit in the interconnection of sequences, functions and equations. The work with sequences leads itself to formulating generalizations and using the algebraic language to express them. This language may, in turn, be used in functions and equations. And equations may again be used to solve problems concerning functions and sequences.

Example 5 – Investigating a polynomial function. The following task concerns the exploration of the properties of polynomial functions. This work was carried out in a grade 10 class by the class teacher, Cristina Fonseca, in cooperation with another teacher, Alexandra Rocha (Rocha & Fonseca, 2005). The task was proposed after the study of the absolute value and quadratic functions.

During some time the students studied the situations posed in the task concerning the properties of the function \( y = ax^3 \), using the graphing calculator. Given their former experience in explorations and investigations, the students worked on a systematic way: they assigned positive values to \( a \) and recorded the behaviour of the function; then they used the same procedure for negative values of \( a \).

The students formulated several conjectures based on the variation of the parameters of the families of 3rd degree polynomial functions:

- It is symmetric! These values move to here and these to here [referring to the symmetry of the functions \( y = ax^3 \) and \( y = -ax^3 \) relative to the \( yy \) axis];
- As \( a \) increases, the graphic moves way from the \( xx \) axis and becomes close of the \( yy \) axis [referring to the behaviour of the function \( y = ax^3 \)];
- In the positive part of the domain, the function \( y = x^3 \) is given by \( y = \frac{1}{2}x^2 \) and in the negative part of the domain is given by \( y = -\frac{1}{2}x^2 \) [the student compares the graphical aspect of the functions];
- The function \( y = x^3 \) goes through the origin of the Cartesian system.

Some groups, after they identified the simplest properties of the families of polynomial functions went on looking for new challenges, transforming their activity in a real investigation. They asked, for example:

- What is the behaviour of polynomial functions such as \( y = x^n \), with \( n \geq 4 \)?
- The points of intersection of the linear and quadratic functions belong to the graphic of the product function of the two functions?
- The 3rd degree polynomial function does cross the points where the linear and quadratic functions intersect the \( xx \) axis?
- What relation exists between the signal of the linear and quadratic functions and the signal of the 3rd degree polynomial function (product of the former two)?
- How to find algebraically the relative extremes of a 3rd degree polynomial function?

The students from one group, after they assigned the value 2 to all the parameters of the quadratic function \( g \) and the linear function \( h \), observed that the 3rd degree polynomial function that is the product of those two “cross the points where the functions \( g \) and \( h \) cross each other”.

To validate their conjecture, the students tested other cases, assigning different values to each of the parameters, \( g(x) = 2x^2 + 3x \) and \( h(x) = 4x + 5 \). Walking by the group, Cristina saw that the students seemed rather unhappy and questioned them:

Cristina – What was your idea?
Ricardo – We gave the same value to all [the parameters]. Then we introduced [in the graphing calculator] the three functions and we saw that the function [3rd degree polynomial] crossed the points were the other two intersected. Now we changed the values and it is not right anymore!
The students had made a conjecture and were quite convinced it was true. They had difficulty in admitting that their conjecture could be false. Using the graphing calculator, they showed the teacher the examples that they had considered. To make them reflect on the information already available, the teacher suggested that they determined the intersection points of the functions, using the facilities of the graphing calculator. The teacher moved way from the group and the students, after new explorations, refined their conjecture:

Ricardo – I know! The function \([3\text{rd degree polynomial}]\) crosses the points were the other functions \([\text{linear and quadratic}]\) intersect the \(xx\) axis.

Patrícia – Then say it!

Ricardo – Wait! I am not sure!

To validate this conjecture, the students determined algebraically the roots of the three functions: \(-1.5, -1.25, 0\). They concluded that the roots of the \(3\text{rd degree polynomial}\) function are the roots of the corresponding quadratic and linear functions. This was a proof for a specific case that, of course, could be generalized to the general case of any \(3\text{rd degree polynomial}\) function, or any polynomial function, or any function that is a product of two given functions. The work of the students, however, took a different direction, seeking to explore other relationships among the three functions, such as, for example, the signal and relative extremes.

When the task was completed, the final discussion took place. This discussion provided new insights, leading sometimes to the formulation of new problems and new conjectures and assigning a high value to justifications and proofs. The dynamic of the discussion led the students to compare their ideas and the others’, to appropriate these ideas, and to put relevant questions, revealing a significant understanding of the topic.

3. Direct teaching and exploratory learning

The examples of the previous section illustrate some key ideas about mathematics teaching and learning that I now address in more general terms.

Tasks. At the core of the former situations there were investigations, explorations and problems. It is important to note the difference between these tasks and exercises. A set of common exercises is the following:

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<th>Simplify:</th>
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<td>a) (\frac{6}{12} = )</td>
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<td>b) (\frac{3 \cdot (10 - 7)}{17 - 2} = )</td>
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<td>c) (\frac{20}{18 - 9} = )</td>
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In an exercise, a computational procedure or a straightforward reasoning provides the answer. Furthermore, the question is clear as well as the given conditions.
A problem may be a task such as: “What is the smallest integer number that, divided by 5, 6 and 7 all yield 3 as remainder?” A problem clearly states what is given and what is asked, but there is no straightforward way to find the solution.

And this is an example of what we may call an investigation:

1. Write the table for 9s, from 1 to 12. Observe the digits in the different columns. Do you notice any pattern?

2. See if you find patterns in the tables of other numbers.

Here the question is somehow open as the reader does not know what kind of “pattern” can be found. Whereas in a problem we begin with a well formulated question, in a investigation, deciding exactly what our question is is the first thing we need to do.

We can differentiate tasks according to two main dimensions: (i) structure, ranging from closed to open, and (ii) complexity, ranging from accessible to complex (figure 3). Explorations and investigations are open tasks but with different complexity. Explorations are most suitable to help with the development of new concepts and representations. Investigations are necessary so that students go through a real mathematical experience of formulating questions, posing and testing conjectures, and arguing and proving statements. Problems are necessary to challenge students with non-trivial mathematics questions. And exercises are important to consolidate students’ knowledge of basic facts and procedures. In consequence, the teacher cannot do his/her job properly using just one kind of task – the issue is to select an appropriate mix, taking into account the students’ needs (Ponte, 2005).

Of course, tasks differ in other dimensions, such as the time needed to do them. For example, investigations that take a long time to complete are usually called “projects”. Another dimension of tasks is pure/applied. In our examples, some tasks were framed in “real-life” contexts (Sousa; Almiro) and others in “pure mathematics” contexts (Segurado; Matos; Rocha & Fonseca).

Classroom roles. Usually, a class in which students work on explorations or investigations has three main segments (Christiansen & Walther, 1986): (i) introduction of the task; (ii) development of the work, and (ii) final discussion and reflection about what was done, its meaning, and new questions to study. In the introduction, the task is negotiated between teacher and students; during the development of the work the students have an opportunity to work by themselves; and the final discussion is a key moment of sharing ideas and institutionalising new mathematical knowledge. The roles of teacher and students change along these three segments. However, at each segment, rather than a one way flow of information, centred on the authority of the teacher, we may have a classroom marked by multiple and complex interactions.
In the former examples tasks were proposed to the students who had to discover strategies to solve them. They designed a strategy to solve the task and had the responsibility of using logical arguments to convince the others of the correctness of their solutions. Therefore, the student has right to a voice, not only to ask clarification questions, but also to defend claims as an intellectual authority. This is a quite different picture from the teaching in which students receive “explanations” from the teacher, who shows “examples” and indicates “how to do things”. When this happens, the teachers and the textbook are the sole authorities in the classroom.

Controlling the class when the students are more agitated, as in the case of Irene, or leaving them to work with large autonomy, as João Almiro did, that is a decision that the teacher needs to take according to the particular situation. However, in all cases presented, the students are assigned a significant role in their mathematical work as a classroom community.

Classroom communication. In a standard mathematics classroom the teacher dominates the discourse, either providing explanations and examples or posing questions and providing immediate feedback. The sequence IRF is well known – the teacher initiates with a question, a student responds and the teacher feedback closes down the issue, confirming or rejecting this response. We must note, however, that not all the questions fall in this pattern. In fact there are many kinds of questions (e.g., focus, confirmatory and inquiry questions) and questioning is one of the main resources that teachers have to lead classroom discourse (Pólya, 1945; Ponte & Serrazina, 2000).

In our examples the students are encouraged to share ideas with their colleagues, often working in groups or pairs. At the end of a significant work, there are discussions with all the class. These are very important moments in which there is negotiation of meanings (Bishop & Goffree, 1986). Different representations may be contrasted and the conventional representations analysed in detail. The proper use of mathematical language is fixed. This is also the moment when the main ideas related to the task are stressed, formalized, and institutionalized as accepted knowledge in the classroom community.

During group work, the kind of communication that is established among students may vary a lot. Sometimes, there is a real exchange of ideas and arguments. In other cases, only one or two students conduct all the work and the others remain silent. The way the teacher interacts with the students of a group is also of great importance. If the teacher does not respond to the students’ questions, these may lose their motivation in the task. If the teacher provides all the answers, he/she takes out all the possible benefit of the task for the students. This means that the teacher has to deal permanently with many dilemmas in conducting the communication in his/her classroom.

Teaching units. Just by itself, a very powerful task does not much. If the students are going to experience some significant mathematics learning, they have to work on a field of problems for some extended period of time (at least for a couple of classes), where they have the opportunity to grasp the non-trivial aspects of the new knowledge, connect it to previous knowledge, and develop new representations and working strategies.

Teachers have to work through teaching units that, on the one hand, provide a journey that supports students’ learning trajectory (Simon, 1999) on a given theme and, on the other hand, support the development of students’ transversal aims for mathematics learning, including their representing, reasoning, establishing connections, problem solving, and communicating capacities. As Witmann (1984) indicates, designing these teaching units, according to careful criteria, is a major task for mathematics education researchers and classroom teachers.
**Summing up.** This consideration of different kinds of tasks, roles and communication patterns provides a characterization of two main styles of mathematics teaching that we find today in classrooms all over the world in different grade levels. We may call them *direct teaching* and *exploratory learning* (figure 4).

<table>
<thead>
<tr>
<th>Direct teaching</th>
<th>Exploratory learning</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tasks</strong></td>
<td><strong>Tasks</strong></td>
</tr>
<tr>
<td>• Standard task: Exercise,</td>
<td>• Variety: Explorations, Investigations, Problems, Projects, Exercises,</td>
</tr>
<tr>
<td>• The situations are artificial,</td>
<td>• The situations are realistic,</td>
</tr>
<tr>
<td>• For each problem there is a strategy and a correct answer.</td>
<td>• Often, there are several strategies to deal with a problem.</td>
</tr>
<tr>
<td><strong>Roles</strong></td>
<td><strong>Roles</strong></td>
</tr>
<tr>
<td>• Students receive “explanations”,</td>
<td>• Students receive tasks to discover strategies to solve them,</td>
</tr>
<tr>
<td>• The teachers and the textbook are the single authorities in the classroom,</td>
<td>• The teacher asks the student to explain and justify his/her reasoning,</td>
</tr>
<tr>
<td>• The teacher shows “examples” so that they learn “how to do things”.</td>
<td>• The student is also an authority.</td>
</tr>
<tr>
<td><strong>Communication</strong></td>
<td><strong>Communication</strong></td>
</tr>
<tr>
<td>• The teacher poses questions and provides immediate feedback (sequence I-R-F).</td>
<td>• Students are encouraged to discuss with colleagues (working in groups or pairs),</td>
</tr>
<tr>
<td>• The student poses “clarification” questions.</td>
<td>• At the end of a significant work, there are discussions with all class,</td>
</tr>
<tr>
<td></td>
<td>• Meanings are negotiated in the classroom.</td>
</tr>
</tbody>
</table>

**Figure 4. Direct teaching and exploratory learning**

**Challenges to teachers**

One must note that a class with exploration and investigation tasks is much more complex to manage than a class based in the exposition of contents and doing exercises, given the impossibility of predicting the proposals and questions that students may pose. In addition, the students do not know how to work on this kind of task and need that the teacher helps them doing such learning. Notwithstanding their difficulties and limitations, this work is essential in a mathematics class that aims educational objectives that go beyond those that are achieved by doing structured activities.

We need to ask what is necessary for a teacher to carry out such exploratory and investigative work in his/her classroom. An analysis of this activity and its contextual requirements leads us to two main areas. The first area concerns the personal relation with mathematical investigations and the second the use of investigations in professional practice.

**Personal relation with mathematical investigations**

1. To have a good notion about what a mathematical exploration/investigation is, how it is carried out, how results are validated (*What is it/How to do it?*)
2. To feel a minimum level of *confidence* and spontaneity in carrying out a mathematical exploration/investigation;
3. To have a general view of mathematics that is not restricted to definitions, procedures and rules, but that values this activity.

Use of investigations in professional practice

4. To know how to select and adapt exploratory and investigative tasks adjusted to the needs of his/her classes;
5. To know how to direct pupils carrying out investigative work in the classroom, in the phases of introduction, development of the work and final discussion;
6. To have confidence in his/her capacity to manage the classroom atmosphere and the relations with pupils to carry out this work;
7. To develop a perspective about his/her role in curriculum management, so that mathematical exploration/investigations, in combination with other tasks, have an adequate role according to the needs of the students.

These are not competencies that teachers develop from one day to the next. The teachers involved in the projects that I mentioned developed professionally for an extended period of time. As important as their projects, was the work in communicating their experiences, writing papers, making conferences and communications in professional meetings. This enabled a deeper look at the experiences that become an important resource for mathematics education, showing the path that curriculum development and change of professional practice may take.

The development of this competence stands on three main elements: collaborating, researching on our own practice, and getting involved with the professional community, beginning at the school level.

Collaborating. Joining together the efforts of several people is a powerful strategy to cope with the problems of professional practice. Several people working together have more ideas, more energy and more strength to overcome obstacles than an individual working alone, and they may build on the diversity of competencies. To do that, of course, they need to adjust to each other, creating an efficient system of collective work. When one of the members of the group is going through a difficult time, he/she receives the support from the others. When a member is really inspired, he/she energizes all the group. Collaboration may develop within a homogeneous or heterogeneous group. A heterogeneous group may involve teachers of different generations, as well as mathematics educators, psychologists, sociologists, etc. A heterogeneous group may experience more difficulty in finding a proper working framework and dynamic, but, when this is achieved, it may become very creative.

Researching our own professional practice. Teachers’ culture has been essentially that of “knowledge transmitters”. The teacher bridges the gap between scholarly knowledge and school students. Today, this is not enough has the defining trait of the professional identity. Teachers, although experts in their subject matter field, are professionals that face very complex problems and need to do research to solve them. This means that teachers need to be able to identify problems, gather information about them, consider all sides of the issues, test solutions, analyse data and interpret results. They have to present their studies to the other members of the profession interested in the same problems. This does not depend so much in learning “research methods” but, most especially in keeping an inquiry stance (Cochran-Smith & Lytle, 1999), in knowing about defining issues and problems, and learning about theoretical notions that help in interpreting data and results.

Investigating is therefore a new element of the teachers’ professional culture. It requires an integrative view of theory and practice as two sides of a single coin – where there is a theory, there is a practice, and vice-versa. In every situation, establishing a dialogue between theory and practice is a major step towards understanding and solving problems.
Involvement with the professional community. Valuing a culture of research among teachers does not depend only on an obstinate individual agency. On the contrary, it requires a fundamental role of the collective stances where teachers carry out their professional activity, especially the schools, pedagogical movements and associative groups. In Portugal there is an important tradition of innovative projects carried out by collaborative groups and sharing experiences in associative settings. What is still missing is reflective and transformative activity at the school level. Teachers who want to change things in their schools need to carry out their own projects within the schools, showing the results to the other teachers, stimulating reflection, creating the need to know more, to experiment, and, hopefully to get other teachers involved in common initiatives.

Conclusion
Mathematical explorations and investigations can be a significant part of the mathematics curriculum. This is because of a number of reasons:

- They constitute an essential part of the mathematician’s work,
- They favour the involvement of the student in work carried out in the mathematics class, indispensable for a significant learning,
- They provide multiple entry points for students with different levels of mathematical competence,
- They stimulate holistic thinking,
- They can be integrated naturally in every part of the curriculum.
- Although related to complex thinking, they reinforce learning elementary concepts.

With greater or lesser emphasis, either mathematical investigations – or key elements of investigating such as conjecturing, testing, and proving – are recommended in the official curricula in many countries around the world (see Ponte, Brocardo, & Oliveira, 2003).

Investigating, teaching, and learning can be seen as an interconnected triangle. The researcher who teaches benefits from the contact with students, as he or she listens to their questions that may challenge his or her theories and methods. The teacher who investigates can use current examples and open problems, making teaching a lively, stimulating activity. And through investigations, the student may become involved in genuine activities of knowledge construction.

Mathematics teachers and teacher educators have interest to investigate their own professional practice, seeking to understand students’ and student teachers’ difficulties, the factors from the social and school contexts that influence them, and the power of teaching strategies to promote qualitative changes in students’ learning. As I argued elsewhere (Ponte, 2001), students may explore and investigate mathematics, teachers and teacher educators may investigate students’ mathematics learning and the conditions that enable it.

In mathematics education there are at present two separate worlds. One is the world of research, as an intellectual elaboration with high rigour but with problematic practical relevance. The other is the world of practice, where problems are felt in a cogent way, but where there is little capacity to theorize and to introduce and sustain innovative solutions. We now have an emerging reality, the world of researching practice. One may expect that it will deal with questions with strong practical relevance, with proper rigor and intellectual elaboration. Working towards such an agenda is a joint task of teachers and teacher educators.
References

Developing creative mathematical activities during lessons of mathematics

This paper presents a research project conducted among mathematics teachers. The aim of the project was to improve the teachers’ ability to develop creative mathematical activities. For that purpose, diagnostic activities, workshops and lessons’ observations were organized. The results show a considerable improvement in the teachers’ ability and their attitude towards mathematical activities.

Introduction

Independent and creative thinking of students is an integral part of mathematics teaching. Contemporary mathematics education follows in the direction of mathematical activities. The assumptions of the PISA exams reflect that tendency (Sułowska & Marciniak, 2004). School mathematics, apart from ready-made knowledge (a set of definitions, theorems and procedures) is mostly a domain of a specific human intellectual activity whose product is ready-made knowledge and the tool used is specific mathematical thinking. Thus, that view of mathematics should be formed among the students (Klakla, 2002; Hejny & Kratochvílova, 2005). Mathematics knowledge is not only the main goal of mathematics education, but should also be a tool which enables a student to engage in mathematical activity. As a result of the work on mathematics lessons a student should learn to work like a mathematician – that is s/he should be able to put hypotheses, notice some regularities and relations, argue, justify, etc. Among these, some creative skills should also be included in that work.

However, the results of our research (Maj, 2006) show that:

– Among the mathematics teachers the knowledge and skills regarding creative mathematical activities are insufficient.
– Among the mathematics teachers it is generally erroneously believed that the creative mathematical activities are developed by themselves during the mathematics lesson and do not require any special didactic endeavours, methods or tools to develop them.
– The mathematics teachers do not have experience and skills of undertaking these activities and what goes after that, they cannot provoke these activities and they cannot organize them in the work with students.

It can be noticed that these results have direct influence on the skills and experience of the students for the whole range of creative mathematical activity. The results of PISA 2003 exam confirm these conclusions: only a few more than 10% of Polish students achieved the results from the fifth and sixth level (on a total of six levels) in the scale of mathematical achievement. Sułowska and Marciniak (2004), in their analysis of the solutions of the tasks, pointed out the weak sides of Polish students:
1) The problem of “the upper quarter” – the best of our students are often weaker than the best students in the rest of the world.

2) Difficulties with independent, creative and abstract thinking.

In this connection there is a need of paying special attention on developing creative mathematical activities and elaborating some methods of instruction for these activities. This view appeared and still appears in the literature of mathematics education (Krygowska, 1985, 1986; Polya, 1975, 1993; Mnich, 1980; Klakla, 1982, 2002; Mason, Burton & Stacey, 2005; da Ponte, 2001).

Mathematics teaching should acquaint the students with all the aspects of mathematical activities so far that it is possible. Particularly, the students should have the opportunity of creative work according to their abilities (Polya, 1975).

The essential condition for the development of the skills needed for different kinds of creative mathematical activities among students is the deep understanding of those issues by the mathematics teachers. To that it is necessary:

– to raise their awareness of the necessity of formation of such activities among their students, and
– to develop their skills of organizing the situations which favour the undertaking of different kinds of such activities.

Only then the teachers would effectively form and develop these activities in their work with the students.

**Theoretical framework**

Mathematical activity of a student is “a work of mind oriented to the formation of concepts and to mathematical reasoning, stimulated by the situations which lead to formulating and solving theoretical and practical problems” (Nowak, 1989).

It is worth to underline two things, namely that the mathematical activity is firstly a work of mind and secondly that it should be stimulated. Therefore, it is not a work of a student which appears in a natural way.

A conception of forming creative mathematical activities was worked out by Klakla (2002) and is based on two elements. The first of them constitutes the distinction between particular kinds of creative mathematical activities, which are present in an essential way in activities of mathematicians. These are:

(a) hypotheses formulation and verification;
(b) transfer of a method (of reasoning or solutions of the problem onto similar, analogous, general, received through elevation of dimension, special or border case issues);
(c) creative receiving, processing and using mathematical information;
(d) discipline and criticism of thinking;
(e) problems’ generation in the process of the method transfer;
(f) problems’ prolonging;
(g) placing the problems in open situations.

The second element of that conception refers to the multistage tasks which:

– consist of series of tasks, problems and didactic situations which have a specific structure,
– are based on problematic situations,
– connect different kinds of creative mathematical activity with each other in complex and rich mathematical-didactic situations,
– provide a specific laboratory of creative mathematical activity for the students.
Ponte (2001) states that there is some similarity between research activity of mathematicians and activity of the students in the classroom.

Of course, there are differences between the knowledge held by both, their degree of specialization, the time they spend, and their relation with the subject. However, their problem-solving activity is of a similar nature. Hadamard (1945), a well-known mathematician, refers to this, for example: “Between the work of the pupil that tries to solve a problem in geometry or algebra and a work of invention [by the mathematician] one can say that there is only a difference of degree, a difference of level, both works being of a similar nature”.

That opinion was also shared by Ernest (1991) and Polya (1975). Polya adds that the students should have an occasion to experiment with many aspects of mathematical activities. Based on examples of some problems he shows that the teachers can create the conditions to make the students develop creative and independent work (Ponte, 2001).

**Methodology**

In this paper we present a short description of the research carried among a group of mathematics teachers and also a description of the lesson conducted by one of the teachers who took part in the research, with the analysis of that lesson from the point of view of creative mathematical activities.

A group of seven teachers of mathematics (of gymnasium and high schools) has taken part in a series of workshops from March to September 2006. The workshops were organized as part of the Professional Development of Teachers Researchers (PDTR) project (226685-CD-1-1-2005–PL-Comenius-C2.1), during the mathematics course. The main content of that course was solving different kind of mathematical problems which were supposed to be challenging for the teachers. The workshops were organized around three multistage tasks. They consist of solving some chosen tasks – open-ended problems and discussing the possibilities of introducing the students to the particular problem. The didactical comments produced were related to the organization of the work with the students and some reflections on different kinds of mathematical activities which the students had the chance to undertake in the process of solving those tasks.

The main aims of the workshops were:

– developing the skills of undertaking creative mathematical activities among mathematics teachers,
– raising the mathematics teachers’ awareness of the need to develop creative mathematical activities and developing the skills of provoking these activities among students,
– showing the teachers a model of working with the students.

Our purpose was to influence the development of the teachers’ skills in organizing situations that – under certain circumstances – can lead to creative mathematical activities which are favourable to be undertaken by the students.

After the end of the workshops the teachers had the task to prepare and conduct a mathematics lesson which main aim was to develop some creative mathematical activities among students. In their previous experience concerning the preparation of the lessons and the determination of the lessons’ aims, the teachers were used to focus on the mathematical content enclosed in the curriculum. Now they had to concentrate on mathematical activities which they will form and develop around a theme of a lesson.

The observations of these planned lessons had the aim to show us if the teacher can plan and organize a work of his/her students in such a way that they can have the opportunity to undertake different kinds of creative mathematical activities. However, it was less important what class that lesson is conducted in and what mathematical content it is related to.
Developing creative mathematical activities during lessons of mathematics

We will present a short description and analysis of the lesson of the teacher who participated in the workshops. That analysis was conducted in the direction of the answers to the following questions:

- Does the teacher develop any mathematical activities among his/her students and if yes, what kind of activities does s/he develop?
- Does s/he stimulate his/her students to undertake creative mathematical activities?
- Do the ways the teacher acts favour independent and creative thinking of the students?

The data collected comprised of audio recording of the lesson (45 min.) and notes. After this a full transcription of the audio recording and analysis of this transcription was made.

**Analysis of the lesson**

The lesson was conducted on 22nd October 2007 in the second class of gymnasium (16 students, 13–14 years old).

The students were working in four groups (3–5 persons each). The lesson started with the teacher posing the following problem (Swoboda, Turanu & Urbanska, 1997).

Tang Ming designs square swimming-pools. Each swimming-pool has a middle part which is also square and filled with water. Around each swimming-pool there is an edge made from the white squares. These are the first two swimming-pools designed by Tang Ming:

![Swimming-pool 1](image1.png)  ![Swimming-pool 2](image2.png)

Then the teacher asked the students: “What would you like to ask about? What questions would you like to ask about this task?”.

The students worked in groups for about 10 minutes and formulated the questions to that situation. Every group wrote in their exercise books from five to eight questions and then presented their lists of the questions. Some of them were the same (I), some of them were not mathematical (II), some appeared only once (III). For example:

(I) What are the areas of the swimming-pools? What are the perimeters of the swimming-pools?
(II) Why Tang Ming does not design round swimming-pools?
(III) How will the next swimming-pools look like? What per cent of the areas are the white squares?

The teacher chose five questions from all of them:
1. How will the next swimming-pools look like?
2. What are the areas of the swimming-pools?
3. What is the difference between the areas of the water and the white tiles?
4. What part of the whole swimming-pool is the middle part?
5. What part of the whole swimming-pool is the edge?

To systematise the work of the students the teacher gave them sheets with the table:

<table>
<thead>
<tr>
<th>Number of the swimming-pool</th>
<th>Number of the white squares (the edge)</th>
<th>Number of the blue squares (water)</th>
<th>Total sum of the white and blue squares</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The students worked in groups for around five minutes and then they shared their own ideas and filled the table. The students filled the first two lines by counting the number of the squares on the drawings. To fill the next lines they either drew the next swimming-pools, or “drew them in the head” imagining (visualising) the next swimming-pools, or they noticed some arithmetical relations between numbers. During the students’ work the teacher did not suggest any ways of solutions, she only motivated them positively and appreciated their exertions:

1. Teacher: /is checking the results in every group/: the third (swimming-pool), the fourth, the fifth, ok, Ela has it already. Do you also have?
2. Some students: We also have the fourth and the fifth.
3. Teacher: You also have the fourth and the fifth? But there are no pictures... the third, fourth and fifth...
4. Kuba: We have just thought up how it would be!
5. Teacher: So? What have you noticed, Kuba? In what way, though there are no pictures, did you calculate the third, fourth and the fifth? What did you observe there?
6. Kuba: That you add the blue one.../he is showing in the straight line/. In the first (swimming-pool) was one blue, then two, three, four etc.
7. Teacher: Ok., Kuba noticed that. We will wait for one moment because I see that you are still working...

The teacher was interested in what way the students calculated the numbers of the squares for the swimming-pools 3, 4, 5, not only the result of that calculation. By asking the question in (5) she suggested the student that he is clever, she motivated him and provoked him to give the explanations. This resulted to the student wanting to share his own idea of creating the next swimming-pools. The student used simple, informal language, referring to the concrete examples, but he also had a general image of the situation, he saw some relations in it. The teacher, by not asking about the result, from the one side encouraged her students in argumentation, from the other side she valued them, because for her their way of thinking was important. She also gave to all students the chance for discovery (statement (7) is evidence of that).

The teacher very often asked the following questions and gave the following instructions:

How did you calculate this? In what way did you calculate this? What did you notice? How did you get it? How come you know about it? Justify what you wrote. What if somebody calculated the numbers in a different way? Why? Are you sure? What is the rule?

The result of these questions was that the students had to justify every number written on the table. It stimulated them to discover arithmetical regularity and then it “forced” them to make generalisations and formulate these rules. Ipso facto they verified their statements, thus developing the skills of communication about mathematics, together with the skills of argumentation and critical thinking:

8. Teacher: Where does the number 16 come from?
9. Radek: plus 4 to the previous one
10. Teacher: Ola, and you? /she goes to the next group/
11. Ola: also plus 4
12. Teacher: And you? /she asks the last group/
13. Ania: also plus 4
14. Teacher: And are you sure that if it is here, for example, plus 4 /she shows the numbers 8 and 12 of white squares for the swimming-pools 1 and 2/, so maybe here it is plus 5, and then it will be plus 6, plus 7,...?
15. Radek: No, because we calculated!
16. Teacher: Did you calculate? Did you check that here it will not be a difference 5?
17. Radek: There has to be plus 4, we calculated those squares!
The students had only two examples, but they formulated a rule. The teacher tried to sow “a grain of uncertainty” (14), but they were convinced about their own thinking, they were independent in thinking (17). The teacher, wanting to provoke them to check and justify their own notices, put a hypothesis that another rule exists. She implied that when you have only two examples it is hard to state that a rule will be also true for the rest cases. Therefore, she gave them another rule which was sensible and could also be true for those two numbers. So the students had to refute that hypothesis.

After filling the first five lines in the table, the teacher asked the students to calculate the values for the tenth swimming-pool.

18. And now I have another question: If you know what rules manage the filling of the next columns and lines, then can you fill the table for the tenth line?
19. Kuba and Jacek raise their hands.
20. Kuba /goes to the blackboard/: I’m substituting here /in the second column – the number of white squares/ a different formula: it is the number of the swimming-pool multiplied by 4 and plus 4.
21. Teacher: Aha, you discovered a different rule, that it is the number of the swimming-pool multiplied by 4 and plus 4… So for this column you discovered two rules… Then try to use the first rule: plus 4 to the previous line… Can we do in that way?
22. Kuba: No, then we would have to calculate in turn every swimming-pool…
23. Teacher: exactly, we would have to calculate in turn every swimming-pool… What about the second rule? What was it?
24. Kuba: the number of the swimming-pool multiplied by 4 and plus 4.
25. Teacher: Is it easier in that way?
26. Kuba: Yes
27. Teacher: So try to do it.

By asking about the tenth swimming-pool (18) the teacher provoked the students to generalise and she “forced” the formulation of the rule for the white squares in another way; instead of “adding 4 to the previous line”, the students proposed “the number of the swimming-pool multiplied by 4 and plus 4”. Kuba was able to pass easily to the different situation, to see some relations and to generalise. However, the teacher wanted the students to convince themselves that the recurrent relation stopped playing its role at that point. So she asked Kuba to use the first rule (21). The student stated that it is easier to use the second rule, because “then we would have to calculate in turn every swimming-pool” (22). Here he used economisation of the processes by using algebraic language.

The lesson ended with the teacher’s request to write the rules of every column in algebraic language or in words (if somebody could not do this symbolically). Some students wrote the rules using algebraic expressions during the lesson.

Conclusions
The choice of the open situation during the lesson let the students pose the questions by themselves. They could show their ability in creative invention. One thing is solving a task which is imposed by the teacher and another thing is to solve a task which was to some extent thought up by yourself. In the second case the engagement of the student in the process of solving the task is usually much bigger, which was what we observed during that lesson. It stimulated them to undertake mathematical activity – to formulate hypotheses, whereas the frequent “how” and “why” questions of the teachers resulted in the necessity of verification, thus enhancing the skills of argumentation and critical thinking. The students proposed the question about the “look” of the next swimming-pools (3, 4 and 5), therefore they prolonged the problem. However, the question of the teacher about the tenth and then the n-th swimming-pool motivated them to
generalize and directed their thinking to the more abstract level, at the same time without suggesting anything (the students noticed that the recurrent relation did not play any role here). During that lesson the students had the opportunity to discover and construct algebraic formulas, describe some general rules, so therefore, they had the opportunity to get the sense of these formulas by discovering recurrent and variable relations.

In our observations we mainly focused on the actions of the teacher during her work with the students. During the described lesson the students had the opportunity to undertake a few kinds of creative mathematical activities. The actions of the teacher were limited to the organization of the teaching process, in order to let the students put questions, formulate problems, and then discover some mathematical relations. The teacher stimulated the work of the students in the direction of creative mathematical activities; she did not impose her own way of thinking, she was flexible and reacted positively to the ideas of her students. And all these favoured students’ independent and creative thinking.

References

The aim of our research was to investigate whether primary school pupils (7–11 year olds) can distinguish between simple arithmetic errors and misconceptions. Data from a specifically designed task are presented. We demonstrate the importance of teachers’ practice in the development of pupils’ reflexive thinking.

Theoretical framework
There is a huge quantity of literature on mathematical misconceptions using a range of different research strategies and analytical techniques (e.g. Zan, 2000; Swan, 2001; D’Amore & Sbaragli, 2005). For example when analysing of misconceptions associated with Stochastic Thinking, Tversky & Kahneman (1983) suggest that they originate from children’s interaction in the classroom or in the physical and social world or from their prior learning. In contrast, Smith et al. (1993) refute the ‘misconception approach’ on empirical and methodological grounds, arguing that misconceptions are in fact context specific and not general and that, for example, they may be due to the absence of appropriate tools to explore the questions. The literature tends to focus on activities for diagnosing and remedying misconceptions. Thus, typically, an adult perspective is adopted and a teacher’s or a researcher’s view of misconceptions is presented. Here we focus on the child: do primary school pupils perceive there to be a difference between mistakes and misconceptions in Arithmetic? We use the term ‘misconceptions’ as proposed by Kaldrimidou & Tzekaki (2006): “to identify the difference between the meaning that students construct about a mathematical concept and the concept itself”; and ‘mistake’ as the term ‘slip’ proposed by Schlöglmann (2007): “Errors that are based on the incorrect application of a formula, or simple mistakes in a calculation”.

The research
This paper stems from a meeting of the first author and some Italian primary teachers. The discussion turned to pupils’ apparent boredom when their teachers revisit incorrect work either with a whole class or with a child individually. This prompted Guastalla to share an activity she designed to encourage pupils to become more involved in the marking process and hence their mathematical understanding. Here we have modified Guastalla’s idea to investigate pupils’ perceptions of what constitutes an arithmetic error as opposed to a mathematical misconception.

1 The authors gratefully acknowledge the support of the British Academy (Grant no. LRG-42447) which provided a platform for this study.
2 This meeting was organized in the School Year 2005–2006 for the aims of the project sponsored by British Academy. Teachers Carla Aleotti, Maria Bertagna, Rossella Guastalla, Maura Previdi, Roberta Santelli and Nunzia Seminerio of Primary school of Viadana (MN) – Italy took active part with the proposal of the item used later with children.
We presented 253 children – aged 7–11 years–old – with a set ‘children’s’ work and asked them to adopt the teacher’s role. The tasks were designed to include correct and incorrect answers with the latter reflecting arithmetical misconceptions rather than simply careless work on the part of the ‘pupil’ who completed them. In reviewing the tasks (see below) teachers were given an opportunity to reflect (a research by-product) on the notion of misconceptions for, unlike practitioners in the U.K., it is not customary for their Italian counterparts to make a clear distinction between mistakes and misconceptions. As a result Italian children are therefore not explicitly exposed to this distinction and only few of them appreciate that there is a difference between mistakes and misconceptions. Interestingly, until relatively recently, little attention has been paid to this seemingly subtle – but important – distinction in the academic literature making it a challenging concept for Italian teachers.

Our hypothesis was that a change in the children’s perspective, from pupil-learners to pupil-teachers, could influence their attitudes. Cobb’s (1985) example of Scenetra showed that her performance improved when she assumed the role of teacher. Moreover as the teacher is – in the pupil’s mind – the person who is closest to scientific contents in the classroom, it is possible that, in this position, the child-teacher could identify mistakes and misconceptions more successfully.

The task

We worked with 253 pupils (52 grade 2, 89 grade 3, 65 grade 4 and 47 grade 5). The tasks for all grades were of the following type:

Look at the task of a pupil of another parallel class. (1) Please mark and assess the work, giving it a grade; (2) Try to explain the line of reasoning s/he applied.

There followed a list of fully completed exercises, some of them correct and others wrong (cf. figure 1 for 4th graders). Teachers adapted the text to their class inserting some of the most typical wrong answers so that they could observe the reactions of their pupils.

The organization of an operation in column is different in different countries. In figure 1 we have used the Italian style of representation, adopted by the Italian teachers. We have, however, translated the original Italian ‘da’ (decina), in the English ‘t’ (tens).

<table>
<thead>
<tr>
<th>Please mark the work of a child of another grade 4 class. You must correct any mistakes and explain why you think they occurred. Please give a grade for the work.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) Place the sign &gt;, &lt; or = in the following pairs</td>
</tr>
<tr>
<td>45 &lt; 54</td>
</tr>
<tr>
<td>Calculate in columns</td>
</tr>
<tr>
<td>2) 17 + 35 =</td>
</tr>
<tr>
<td>17 +</td>
</tr>
<tr>
<td>35</td>
</tr>
<tr>
<td>412</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>3) 3,5 + 4,2 =</td>
</tr>
<tr>
<td>3,5 +</td>
</tr>
<tr>
<td>4,2</td>
</tr>
<tr>
<td>7,7</td>
</tr>
</tbody>
</table>

Figure 1. The fictitious 4th grader’s task
During the preparation of the task, teachers confronted the problem of distinguishing between mistakes and misconceptions as they had to identify the most appropriate examples for the research and, in doing so, consider the most frequent mistakes/misconceptions which occurred in their classrooms.

As specified above, all of the wrong answers presented in the tasks could be explained in terms of arithmetical misconceptions. They also had to be relatively simple so that they were not beyond the pupils’ intellectual capabilities and yet rich enough to reveal children’s understanding. As such the tasks were designed to provide teachers with the opportunity to clarify how learners face this kind of activity (Schubauer-Leoni et al., 2004) as the children explain their ‘colleague’s’ reasoning for incorrect solutions.

Results

The task proved successful in every class: each wrong exercise had at least one pupil suggesting that it was right and for each right exercise there was at least one child stating that it was wrong, supporting the statement of Santi & Sbaragli (2007) that misconceptions are unavoidable and, as stated in David et al. (1994), they have a constant presence.

Some pupils took the task very seriously. Others did not. In all cases, however, the children provided some interesting insights into their interests, their involvement with mathematics and their understanding of the subject. We generally found three levels of response:

– The first is the detection of the wrong answers and suggests that the child has an awareness of algorithms. In terms of Bruner’s work (1992) this level requires the logical-scientific thinking related to what it is to do mathematics.

– The task also asks for explanation of the thinking of fictitious colleague. This is a second level and not all of the pupils reached it. This level requires narrative thinking – as defined by Bruner (1992) – in order to produce a plausible report. This sort of thinking is related to one’s ability to talk about mathematics.

– The third level is the pupils’ assessment of the task.

A few children went beyond the task requirements and we found what might be termed a fourth level as they evaluated the task and its goals. For example:

“Goals [of the task]: To know the use of <, > or =; to know computation with natural numbers and with decimal numbers, but the joint aim is to recognize the (place) value of digits.” (4 – A.B.)

We could not find any references in the literature on the third and fourth levels thinking we observed.

Analysis of L.’s protocol

The first three levels are evident in the protocol of L., a grade 2 pupil (figure 2). As far as possible the following translation from Italian reflects L.’s expressions:

– she corrects the incorrect answer 37 < 29 saying: “NO you got mixed up”; 
– she comments on the correct response 23 ≥ 22 with “Yes”; 
– she incorrectly marks a correct answer 45 < 54 saying: “Not true as 54 is not > than 55”; 
– she corrects 17 + 35 = 412 with “52” adding, “NO PR. you counted too much. Next time do it this way.”
– She also presents a different method for the same computation where she singles out the amount carried.
– She marks 24 + 50 = 70 saying: “NO you did a computation mistake”.
– Finally she grades the task as “Distinguished +”.

The words “Si” (Yes) and “No” (No) indicate the first level (described above) in which L., employing logical-scientific thinking, states the exactness of her ‘colleague’s’ answer. The sign “>” in the first exercise and “52” and “4” in the two addition items belong to the same level.

The second level is evident in her comments. It is perhaps surprising that L. deems that the incorrect answer to the first question was the result of ‘mixed up’ thinking and yet she does not specify the nature of the confusion. Comparing this to L.’s answer to the third question (i.e. 45 < 54), however, suggests that this sort of muddle is familiar to L. With a similar task (i.e. 189 > 198) some grade 4 pupils suggested that confusion is the result of a faulty consideration of place value (misconception), since it could be explained on the basis of comparing the unit digits.

L.’s comment, “you counted too much” to the solution 17 + 35 = 412 suggests that, even if she is able to apply the standard algorithm of addition in columns, she is also aware of the counting-on technique as she wrote ‘count’ instead of ‘add’. By this she could have meant, ‘you have gone too far’ or ‘your result is too big’. G. (also grade 2) wrote something similar which can be translated from Italian in one of two ways: “He made a mistake, because he counted incorrectly and also he was in a hurry” or “He made a mistake, in my opinion, because he counted too much, badly and also he was in a hurry”. Such reference to counting was still evident in three grade 5 responses suggesting that awareness of counting-on may be a strategy for testing the correctness of an addition calculation for some pupils throughout the primary years.

The two letters “PR.” could mean “Prova” (Experience). The girl was interviewed about this, but she did not remember what she meant. L.’s comment on the exercise 24 + 50 = 70 – “NO you did a computation mistake” – is attributable to the second level, comparing her remark to the previous one, shows an evident difference in her thinking.

The third level is exemplified by L.’s mark “Distinguished +”. In Italy grades are not expressed by numerical or literal values. The idea is to give an assessment with formative features. In practice this laudable aim is substituted by a ‘scale’ in which a task is assessed by an adjective: insufficient, sufficient, good, distinguished, excellent. These grades might be modified by adverbs, such as: seriously (insufficient), only just (sufficient), nearly or fully (sufficient, distinguished, good, excellent), and so on, according to teachers’ linguistic creativity. Teachers may further refine grades adding the signs “+” or “–”.

Other findings
Sometimes in mathematics it is sufficient to estimate an approximate answer, without specifying the exact result. Some children worked effectively in this way and demonstrated a good awareness of the properties of operations.

“since 17 + 35 makes a number of two [digits]” (M.B. Grade 2)
“the result 412 is impossible since the numbers are too small.” (L. & E. Grade 4)
Teaching practices revealed through arithmetic misconceptions

“over the nine there was nothing and it is as he was putting 0 and then he computed 0–9 and this cannot be done”. (A. & R., Grade 4, commenting 202 – 96 = 106)

“the number is too small for the high class of thousands” (A.P., Grade 5, commenting on the task 27030 : 3 = 91)

To conclude thus far; asking pupils to take on the role of teacher looks like a suitable didactic device for exploring their arithmetical misconceptions and their ability to assess the correctness of others’ solutions. In such circumstances pupils show, “a different kind of mathematics that is often intolerably hard” (Gray & Tall, 1994, p. 116).

Moving on to the second level – explaining another’s thinking – Table 1 shows evidence of this second level across the grades studied.

<table>
<thead>
<tr>
<th>Grade 2 (No. 52)</th>
<th>Grade 3 (No. 89)</th>
<th>Grade 4 groups (No. 19)</th>
<th>Grade 4 singly (No. 22)</th>
<th>Grade 5A (No. 23)</th>
<th>Grade 5B (No. 24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>36.54</td>
<td>36.54</td>
<td>36.54</td>
<td>36.54</td>
<td>36.54</td>
<td>36.54</td>
</tr>
</tbody>
</table>

Table 1. Presence of second level (in percentage)

The children reacted to the presence of wrong answers in different ways. We would suggest that this is likely to depend both on the teachers’ practice, and the number of years of schooling. Significantly Table 1 shows that pupils in grade 4 were more successful than any other group but, given that their teacher, Rossella, usually requires this second level activity, perhaps it is not surprising. Indeed we found some of these 4th graders – working either singly or in groups – to be very detailed in their comments of wrong results. This was particularly true in the case of the wrong answers to inequality exercises.

Having said that, we obtained protocols from all of the classes showing sensitivity to misconceptions. Clearly we did not expect children to use the word ‘misconception’ although we found some explicit instances of ‘reasoning mistake’ versus ‘slip of pen’ among grade 4 pupils and we interpreted the former as the child’s term for ‘misconception’. In order to provide a more detailed example of such answers, we consider the following exercise given in Class 5A (N = 23): 12506 + 99 =12407. We have included the number of children responding in a particular way in brackets:

1) the addition sign is inappropriate since the result is obtained by subtraction (3);
2) the addition calculation is wrong (19), with:
   2.1) (global) attention to the whole calculation modifying the hundreds digits, the tens and the units, eventually with right result (3);
   2.2) (local) partial attention to the calculation restricted to the tens and units digits (1)
   2.3) (local) only attention to the units digit (5)
   2.4) simply the statement that the result is wrong (7)
   2.5) the child purposes another wrong result (1)
3) the addition calculation is considered right (3).

All children stating that their ‘colleague’ had given the wrong result are correct, but closer examination shows that – despite being in the same class – their reasoning differs. For example, some demonstrated a global understanding of this task and, in the case of (1) above, considered the misconception to be due to the wrong interpretation of the addition sign, while those who responded as in (2.1) perceived it to be the result of the incomplete understanding of how to use the addition algorithm. In both cases we can assess these answers as indicating the perception of the presence of a misconception.
The ‘local’ pupils (2.2 and 2.3) show an attitude that can be defined by Gardner (1993), (quoted in Zan (2005)), as the ‘compromise of correct answer’: many teachers and pupils consider that the education act is successful when learner succeeds in giving answers that teacher assesses as right. Here the change of perspective transforms the ‘compromise’ in giving the answer that a colleague is wrong, without studying the reason behind the error in depth.

From another perspective many of the children’s responses demonstrating second level answers to our question appeared to be strongly influenced by their own classroom experience and appeared to mirror teachers’ typical reactions to mistakes, albeit, in some cases, inappropriately. We observed only three pupils giving type (1) responses speaking with their own voices.

Focusing on the third level: sometimes there was a great distance between the assessment and the mark. An anonymous 3rd grader wrote, “You mistook everything, you are not able to do….You disappoint me – Excellent – –”. We might judge this assessment, in itself, severe, but the mark expresses the child’s own hope. There is also a moral blackmail in the judgement. This distance was apparent in the majority of the protocols and many of them illustrate the affective dimension with which pupils understand the mark.

Conclusions

We believe that teachers should combine the teaching of algorithms with reflection of the mathematical activity for, as stated by, Haylock & Cockburn (2003), “the learning of recipes for answering various types of questions is not the basis of understanding in mathematics” (p.1). This reflective activity could fill the gap between the teacher’s assessment and the child’s understanding of the assessment. Indeed such sharing between teacher and learner could enhance the learning environment. Our data, for example, suggest that the second level activity prepares a fertile soil for understanding the difference between misconceptions and mistakes and thus broadens adults’ and children’s appreciation of mathematical concepts.

References

Teaching practices revealed through arithmetic misconceptions

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Students discussing their mathematical ideas:
Group-tests and mind-maps

In an explorative research project, teachers experimented with new ideas to make their students discuss (i.e. show, explain, justify and reconstruct their work) their mathematical ideas with each other. Two kind of special tasks were developed: group tests and mind maps. Also, the role of the teacher was studied in order to evoke discussions between students. In this workshop, tasks and results will be presented, experienced and discussed.

This paper investigates two examples of how to evoke mathematical discussions between students. With mathematical discussions we mean discussions in which students show each other their mathematical (thinking) work, explain it to each other, justify it and reconstruct their (thinking) work, described as key activities in the process model of Dekker & Elshout-Mohr (1998). In a new Amsterdam school for secondary education (13 and 14-year-old students), a teacher, a student-teacher and a researcher have been experimenting with two types of tasks for small-group discussions in the mathematics lesson: group tests and making mind-maps.

**Group tests**

Students have been working in groups of three on a test on the subject of ‘unities’. The students had prepared for the test individually and were grouped by the teacher on basis of their expected level and preparation time. Groups were not allowed to interact, but within the group, students were encouraged to discuss with each other. Eight classes were making their test in this way. In each class, one group was audio recorded. Both teacher and students were enthusiastic about this way of working. Students mentioned that they were better able to think, since they could discuss their ideas with peers. Moreover, some students were surprised by the experience of collaborating with a peer with whom they had never been working before. It appeared that within most groups, students performed key-activities. Retrospective analysis made clear what factors determined the success of this method (group size, preparation at home, assessment, group composition).

**Mind-maps**

A mind map is a diagram used to represent words, ideas, tasks or other items linked to and arranged radially around a central key word or idea. It is used to generate, visualize, structure and classify ideas, and as an aid in study, organization, problem solving, decision making, and writing. In groups of three or four, students made a mind map starting with ‘measurements and unities’, in order to evaluate and structure what they learned on this subject. The mind-maps of
Students discussing their mathematical ideas: Group-tests and mind-maps

the groups were shared with the whole class and a joint map was created on the white-board. For the next lesson, each students prepared a card which represented a word from the joint mind-map, or a picture related to this. During a whole-class discussion, the cards were structured on a magnetic board and a definitive version of the joint min-map was created. This map was copied on a small poster for all students. During the lessons, audio recordings were made from all groups and a video recording from the overall classroom. The students mind-maps were collected. The audio recordings were analysed on the occurrence of key-activities.

References

Developing teachers' subject didactic competence:
Case of problem posing

Problem posing is discussed as one of the ways of empowering subject didactical competence (pedagogical content knowledge). The work with students and/or teacher can start with examples of problems posed by student teachers and elementary teachers.

Introduction
The programme declaration of the 53rd CIEAEM conference stated that the society had become increasingly mathematical and noted that consequently, the importance of mathematical education and the need to acquire mathematical literacy had grown. This puts great demands on teachers’ professional competence.

In our previous research, especially the need for a good level of subject didactical competence appeared very strongly, i.e., the knowledge of mathematical content and its didactic elaboration as well as its realisation in school practice (Tichá, Hošpesová, 2006). (Let us be reminded of Shulman’s idea: if teaching should become a profession, it is necessary to aim at creating a knowledge base for teaching which encapsulates, in particular, subject-matter content knowledge, pedagogical content knowledge, and curriculum knowledge (Shulman, 1986)).

Many teacher educators (Silver, Cai, 1996; English, 1997; Pittalis et al., 2004) emphasise that the core of mathematical education (in school) is not only problem solving but also (particularly) problem posing. Problem posing was specified by Silver as generating new problems and questions (growing from some mathematical or “non-mathematical” situation; Koman, Tichá 1998) as well as the reformulation of given problem, e.g., by “What if (not)?” questions, by releasing parameters, etc.

Problem posing contributes to the development of students’ ability to solve problems. The ability to formulate questions and pose problems should therefore be an important component of teachers’ competence (knowledge base for teaching). In its development, we see both an aim and a tool for the education of student teachers and teachers. An analysis of posed problems is also a good diagnostic tool; it allows not only to teachers in practice but also to researchers and to teacher educators, to determine the level of understanding as well as causes of mistakes and errors of the student. It is possible to get a lot of didactically interesting information from posed problems (Silver, Cai, 1996; Tichá, 2003). It also enables participants’ to realize deficiencies in their own knowledge which leads to the improvement of their competence.

Example of the workshop
The basis of the workshop could be examples of student teachers’ work (posed problems and their analysis and assessment).
We can start with the exchange of ideas concerning the role and benefit of problem posing. The core of the work will be a gradual discussion on

• the examples of problems posed especially by student teachers,
• the assessment of posed problems,
• student teachers’ analysis of posed problems.

An example of a task for student teachers

14 year old students should solve the following task: Create such a word problem which can be solved by a mere calculation $2/3 \times 1/4$.

One student formulated the following three problems.

1. There were $2/3$ of the cake on the table. David ate $1/4$ of the $2/3$ of the cake. How much cake did he eat?
2. There were $2/3$ kg of oranges on the table. Veronika ate $1/4$ kg. How many of oranges were left (kg)?
3. $2/3$ of the glass was full. Gabriel drank $1/4$. What part of the glass was full?

Which answer (problem) is correct? And, in fact, is any of them correct?

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References

Thinking to the future: Prospective teachers encouraging children's mathematical thinking
Example of a workshop

This workshop is a practical exploration into the lessons of two prospective primary teachers half way through their teacher preparation course. The aim is to develop the quality of their education and, in turn, enhance the mathematical experience and prospects of children in the future.

Background
Recent research has demonstrated that mathematical misconceptions may originate during the earliest years of schooling and yet remain undetected until children encounter more complex mathematical ideas in secondary school or even university (Cockburn and Littler, 2008). There is also a growing understanding of the characteristics of successful teachers (Shulman, 1986; Ball, Bass and Hill, 2004) and, in particular, how they can open up opportunities for advanced mathematical thinking with children as young as 5–6 years-old (Iannone and Cockburn, 2006).

Our knowledge as to how teachers become experts is less well developed, however, Rowland, Huckstep and Thwaites’ (2005) ‘knowledge quartet’ provides a detailed account of the key attributes we might expect of elementary teachers towards the end of their teacher education courses.

This workshop begins with trainee teachers half way through their training and is designed to encourage us to reflect on (a) our own pedagogical practice and (b) how we can encourage future teachers to open up children’s opportunities for mathematical thinking while reducing the likelihood of misconceptions arising in the process. The catalysts for discussion arise from a small observational study of 8 volunteers as they teach mathematics to primary pupils in Norfolk, U.K.

Workshop outline
Introduction: Following a brief introduction to the literature and the research, the group will be introduced to Jodie – a student who is half way through her teacher education course – and the first few minutes of a lesson she taught to 5–6 year-olds on permutations (10 minutes)

Task 1: In small groups colleagues will discuss how they would have continued Jodie’s session had they been asked to take over. (15 minutes)

Timings are very approximate and, should the discussion become particularly lively, adjustments can be made and, for example, task 4 may be omitted.
**Task 2:** The same groups will compare and contrast their suggestions with what Jodie actually did noting, in particular, opportunities to motivate and extend children’s mathematical thinking and any potential for the birth of mathematical misconceptions (*15 minutes*)

**Task 3 and 4:** Repeat the above processes focusing on a second prospective teacher – Morag – working on combinations with 9–10 year-olds (*25 minutes*)

**Discussion and conclusion:** A whole group discussion highlighting common themes and idiosyncratic inspirations ending with any implications for both teacher education and, indeed our own practice, as a result of the workshop (*15 minutes*)

**References**