Increasing teachers’ awareness and skills of generalization
PROSPECTIVE TEACHERS’ MATHEMATICAL KNOWLEDGE OF FRACTIONS AND THEIR INTERPRETATION OF THE PART-WHOLE REPRESENTATION

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Fractions are one of the more complex mathematical concepts children encounter in their schooling. While the majority of existing research addressing fractions has focused mainly on students, leaving aside the teachers’ role and the importance of teachers’ knowledge in and for teaching, we focus on early years’ prospective teachers’ knowledge on fractions and the role of the whole and its (possible) impact in preventing pupils from achieving a full understanding of the topic.

INTRODUCTION/SOME MOTIVATIONAL WORDS

The International Summit on the Teaching Profession has addressed the challenge to equip all, instead of just some, teachers for effective learning in the 21st century (OECD, 2011, p. 5). This requires an emphasis on, among many aspects, “the kind of initial education recruits obtain before they start their job” (ibid). Several studies have documented that teachers have a greater impact than any other factor on student achievement (e.g. class size, school size, or school system) (e.g., Nye, Konstantopoulos & Hedges, 2004). There has been an increasing amount of attention and focus laid on teachers’ knowledge, and on how gaps in such knowledge relate to limited treatment in subject courses prospective teachers’ receive in their education. Studies have shown that an exclusive focus on content knowledge, by increasing requirements for more advanced mathematical courses, has no positive effect on student’s achievements (Begle, 1972). Also, due to Shulman’s (1986) distinction between subject matter knowledge (SMK) and pedagogical content knowledge (PCK), the importance of teachers’ knowledge has received increasing attention.

In mathematics education, Shulman’s ideas were developed further into a framework for teachers’ mathematical knowledge for teaching (MKT) by a group of researchers lead by Deborah Ball at the University of Michigan (e.g., Ball, Lubienski & Newborn, 2001; Ball, Thames & Phelps, 2008). The Michigan group has identified a number of specific challenges related to teaching mathematics, and it is assumed that these challenges (tasks of
teaching), are similar in different countries (Ball et al., 2008). Examples of tasks of teaching are recognizing what is involved in using a particular representation, and linking representations to underlying ideas and to other representations.

To improve practice and teacher training at all educational levels, teacher education has to focus more on teachers’ knowledge, on the tasks involved in teaching, and on the mathematical critical situations and topics identified, which will contribute to a smoother transition of students between educational levels. One of these critical topics concerns fractions. Students struggle to understand both the mathematics embedded, and the different interpretations and representations fractions can assume, i.e. part-whole, quotients, measures, ratio, rate, and operators (Behr, Lesh, Post & Silver, 1983). For such an understanding it is of fundamental importance that a good understanding of the relationship of the parts and the whole, and the possible different “kinds of whole”, be acquired. Students’ limited understanding might be related to how their teachers’ understand and interpret fractions, and such limitations may result from the fact that this topic is not addressed explicitly and does not have the focus that is needed in teachers’ education.

Teachers’ training has in certain respects been left behind in the research. We still know little about how (prospective) teachers’ knowledge of fractions influences students’ broader view of mathematics, and its connection and evolution within and along schooling. This has motivated our research. By calling attention to prospective teachers’ training and the role it has on teachers’ professional knowledge and development, we hope to better understand the (possible) impact such knowledge has on their (future) practices, and on their students’ achievement. With this in mind, we address the following research question:

What kind of subject matter knowledge (in terms of MKT) is revealed about their interpretation of fractions and the role of the whole by early years prospective teachers’, and how can we characterize such knowledge in order to specify critical aspects to focus on teachers’ training?

THEORETICAL FRAMEWORK

Teachers’ knowledge can be perceived from different perspectives. Grounded in Shulman’s (1986) work, some new conceptualizations on mathematics teachers’ knowledge have emerged (e.g., Rowland, Huckstep and Thwaites, 2005; Davis & Simmt, 2006; Hill, Rowan & Ball, 2005). In our focus on teachers’ knowledge, we focus on the MKT conceptualization with its various sub-domains (Ball et al., 2008). One reason for favoring this conceptualization of knowledge is that we perceive the sub-domains of MKT (see Ball et al., 2008) as

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1 We assume teachers’ professional development to start, explicitly, and in a formal way, in prospective teachers’ training and thus this is (for excellence) the starting point for discussing, promoting and elaborating teachers’ knowledge in order to allow them to teach with and for understanding.
a relevant starting point for designing tasks for the mathematical preparation of teachers, and for doing research on what inputs to teachers training shows effects on students and practices. Interestingly, the Michigan group has found a connection between teachers’ MKT, as measured by their MKT items, and students’ achievement in mathematics (e.g., Hill et al., 2005).

Figure 1: Domains of MKT (Ball et al., 2008, p. 403)

The MKT conceptualization of teacher knowledge comprises Shulman’s domains (SMK and PCK) and considers each one of them as being composed of three sub-domains. We will here approach only the sub-domains concerning SMK. SMK comprises what is termed “common content knowledge” (CCK), “specialized content knowledge” (SCK), and “horizon content knowledge” (HCK). CCK is knowledge that is used in the work of teaching, but also commonly used in other professions that use mathematics. It can be seen as an individual’s knowing the topic for themselves – e.g. knowing how to obtain the correct answer when multiplying fractions. Teachers (obviously) need to know how to do this, but it is also common knowledge within a variety of other professions. However, in order to give students’ opportunities to achieve a deeper understanding of the topics (here fractions), besides knowing how to perform the calculations (find the correct result or identify incorrect answers), teachers’ need to know the mathematical hows and whys behind such calculations. Such knowledge on the hows and whys related with fractions is a core knowledge in order to allow teachers’ to (amongst others) being able to explain it to students’, listen to their explanations, understand their work, and choose useful representations of fractions that can support students’ learning. This is knowledge that requires additional mathematical insight and understanding (Ball, Hill & Bass, 2005), and is considered SCK. The last sub-domain is termed HCK, which is described as “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al. 2008, p. 403), and is important for developing students’ connectedness in mathematical understanding along the schooling.

Teachers’ knowledge and what concerns the specificity of the topic being approached (mathematics) is inter-related, it influence and is influenced by a large span of dimensions and aspects. Examples of these dimensions and aspects are teachers’ role, actions and goals (Ribeiro, Carrillo & Monteiro,
Teacher’s participation in professional development programs can contribute to an important part on their awareness of practice (Muñoz-Catalan, Carrillo & Climent, 2006). It also contributes to the development of their MKT and on their awareness of the role of teachers’ professional knowledge dimensions in practice (Ribeiro et al., 2009). We assume that teachers’ professional development starts, explicitly and in a formal way, in pre-service teachers’ education, and thus, this is (should be)2 the starting point for discussing, promoting and elaborating teachers’ knowledge allowing them to teach with and for understanding.

Within the new Portuguese National Curriculum (Ponte et al., 2007), the understanding, representation and interpretation of fractions is transversal to all the first nine years of schooling. In this new curriculum, it is mentioned that the approach to rational numbers should start on the first two years of schooling, in an intuitive manner. Thereafter, one should progressively introduce the representation of fractions, using simple examples. In years three and four, the different interpretations of fractions should be deepen, starting from situations involving equitable sharing or measuring, refining the unit of measure – using discrete and continuous quantities.

Discussing the importance of the role of the whole is a core element in allowing for understanding of all the different interpretations and representations of fractions (Kieren, 1976), and is perceived as a “prerequisite” for such understanding (Ribeiro, in preparation).

Fractions are among the most complex mathematical concepts that children encounter in their years in primary education (Newstead & Murray, 1998). These difficulties can be originated from the fact that fractions comprise a multifaceted construct (e.g., Kieren, 1995) or they can be conceived as being grounded in the instructional approaches employed to teach fractions (Behr et al., 1993). These identified difficulties illustrate the importance of improving teachers’ initial training. A consequence of such an improvement will be increase students’ CCK concerning fractions, contributing to a new and better direction at all educational levels.

METHODOLOGY AND CONTEXT

This paper is grounded in data gathered from an exploratory study between sixty prospective early years’ mathematics teacher in Portugal. By combining a qualitative methodology and an instrumental case study, we focus on these prospective teachers’ MKT on fractions, and on their revealed understanding about the role of the whole.

2 We consider that pre-service teachers’ training should start to assume a broader and important role in teachers training, as it is the first stage and contact with most of the aspects referred to in literature as being problematic and in need of a change.
Data is from a sequence of tasks assigned to these prospective teachers in the context of a course focusing on the SMK sub domains of MKT (with 28 hours of classes, meaning 2.5 ECTS). Fractions were one of many topics approached in the course. Tasks used in the assignment were taken from Monteiro and Pinto (2007), and then modified for implementation in teachers’ training and aligned with the Portuguese National Curriculum for the first nine years of schooling.

Besides focusing on CCK, the aim was also to look into the different interpretations and representations of fractions, in particular the role of the whole. All tasks were discussed in groups of four or five prospective teachers, and at the end there was a large group discussion aiming to obtain a deeper understanding of their knowledge of fractions (SCK and HCK).

The assigned set of tasks was designed with a specific goal to promote the development of prospective teachers SMK (Ribeiro, in preparation) on fractions. They were specifically related to the work of teaching mathematics and they were grounded in tasks of teaching (Ball et al., 2008). In this paper, we only present part of the first task:

Teacher Maria wants to explore with her year one students some notions concerning fractions. For such she has prepared a sequence of tasks involving 5 chocolate bars. What amount of chocolate would 6 children get if we share the 5 bars equally among them?

The prospective teachers were asked to solve the task with two different perspectives in mind: 1) as if they were year one students, and 2) giving their own answer as prospective teachers. In both answers they were supposed to describe and justify what they did and why they did it.

In the analysis we focus on prospective teachers’ mathematical critical situations: their revealed gaps in knowledge, their different interpretations of fractions, and on the role of the whole. Our aim is to obtain a deeper understanding of the mathematical reasons why such gaps occur, in order to be able to design materials to improve teachers’ training and the ways in which we, as teacher educators, approach such training.

SOME RESULTS AND DISCUSSION

Here we present, analyze and discuss answers from some of the prospective teachers. All groups presented at least one numerically correct answer, frequently found by using different ways of dividing the chocolate bars aiming to express the final result as a sum of different numerical fractions. They commonly had difficulty in explaining the sense of the answers.

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3 In Ribeiro (in preparation) the nature of such tasks and of the specificities associated to the context in which they are aimed, are discussed. Part of such discussion concerns, also, the kind of necessary changes to be implemented to tasks prepared to be implemented with pupils/students, in order to contribute to develop teachers MKT in all its sub domains of SMK.
Many of these prospective teachers failed to consider the role of the whole when solving the task, they failed to consider how they would divide the chocolate bars (the whole) in order to “share the 5 bars equally”, and often they did not even consider the importance of finding the “exact” amount of chocolate each child would get. They used either exclusively pictorial representations or tried to represent in different ways (one of) the correct answers. When using exclusively pictorial representations the answers can be divided in two groups: (i) the whole is 5 chocolate bars isolated (simply pictorial answer); the prospective teachers’ then just draws the chocolate bars and divides each part they are obtaining (in each step) in halves and thirds (Pictures A and B); and (ii) the whole corresponds to a continuous unit compose by the 5 chocolate bars; the teacher then draws a representation of the 5 chocolate bars (as a whole) (Picture C).

Figure 2: Examples of student work using exclusively pictorial representations
When they tried to represent (one of) the correct answers in different ways, prospective teacher considers the whole to be the set of 5 chocolate bars, but seen as discrete “subunits”, and their focus was on obtaining the answer using different ways of dividing the 5 chocolate bars. Then, they tried to match the drawings with the numerical representation. For such, they tried to present the same answer throughout the matching of various hypothetical representations with formal fractions notations (Picture D).

Figure 3: Examples of student work considering different ways of representing one of the correct answers

Although they consider different ways of representing one of the correct answers (5/6), visually and algebraically, they did not pay any attention to the different whole in this situation, nor the different notion the whole could take. Their answer would typically be something like: “each student will get exactly 5/6 of the total amount of chocolate or 5/6 of each chocolate bar”.

The prospective teachers who presented different approaches to the answer (which occurred in more than half of the groups) frequently believed that the
reasoning would necessarily be different whenever their way of dividing and representing the solutions algebraically was different. From another point of view, they consider it to be the same to say that the pupil will get: 5 pieces of chocolate; one bar of chocolate (when collecting 1/6 of each bar and transforming it in one other bar with 5/6); or 5/6 of each chocolate bar. Such difficulties in understanding the role of the whole impeded them from being able to interpret, afterwards, different representations and interpretations of fractions in the subsequent tasks.

SUMMARY AND IMPLICATIONS FOR MATHEMATICS TEACHERS’ TRAINING

The subject matter knowledge revealed by these prospective teachers’ is aligned with the knowledge revealed by early years’ students (Monteiro, Pinto & Figueiredo, 2005). Their different ways of seeing a discrete whole, and giving answers involving fractions (and, necessarily, the impact of this on the interpretation of fractions) is problematic, because they show some of the same gaps in knowledge as the ones their (future) students are struggling with.

These gaps in knowledge, which may be admissible at an early stage at primary school level, would make it impossible for them (at least at the time) to develop a broader understanding on the interpretations and representations of fractions. This would limit the learning opportunities they are able to provide to their students, the nature and richness of the tasks they would propose, and these gaps in knowledge should thus become an explicit focus of training.

These results, in terms of the gaps in prospective teachers’ knowledge and the way(s) they consider the whole and, consequently, the notion(s) of fractions, appear problematic to us because the large amount of research being done on fractions (focusing on the students) seems to have had no significant impact in teachers’ training. This led us to problematize our own practice and the focus of the training we are offering, and we began to think differently about teachers’ training, primarily in the direction of reinforcing the primary role of the SMK sub-domains for improving training. This will allow prospective teachers to approach the topics with and for understanding, and with a sense of the possibilities of conceptualizing all the possible levels of generalization. Only through such a change will it be possible to allow students to achieve a global view and understanding of the mathematical topics, and on the ways they relate and evolve along schooling and the different connections between each of them.

It will also allow them to generalize with sense and effective knowledge.

Our final thoughts are introspective reflections as teachers and mathematics educators, informed with responsibility in teachers’ training (at all its different stages). The fact that these prospective teachers reveal gaps in fundamental knowledge is also our fault, and we have to really reflect on this and change both the nature and focus of our training and of the tasks we use in teachers’
training. Such a change would make sense if it really takes consideration of an effective approach between theory and practice, focusing on the specialized knowledge for the mathematical topics, assuming such knowledge to be something that can be effectively taught (Hill & Ball, 2004).

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References


COMPARISON OF COMPETENCES IN INDUCTIVE REASONING BETWEEN PRIMARY TEACHER STUDENTS AND MATHEMATICS TEACHER STUDENTS

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Inductive reasoning is part of the discovery process, whereby the observation of special cases leads one to suspect very strongly (though not know with absolute logical certainty) that some general principle is true. It is used as a strategy in teaching basic mathematical concepts, as well as in problem solving situations. In the paper the results of the study on primary teacher students’ and mathematics teacher students’ competences in inductive reasoning are presented. The students were posed a mathematical problem which enabled them to use inductive reasoning in order to reach the solution and make generalizations. Their results were analysed from the perspective of the problem solving depth and from the perspective of the applied strategies. We also analysed the relationship between the depth and the strategy of problem solving and established that not all strategies were equally effective at searching for problem generalizations.

INTRODUCTION

In many cases the researchers related the inductive reasoning process to the problem solving context (e. g. Christou & Papageorgiou, 2007; Küchemann & Hoyles, 2005; Stacey, 1989). These examinations pay attention to the cognitive process, as well as to the general strategies, that students use to solve the posed problems. Problem solving fosters in mathematics education various kinds of reasoning, more specifically, inductive reasoning.

In literature terminology of various kinds is used when addressing reasoning in mathematics: deductive reasoning, inductive reasoning, mathematical induction, inductive inferring, reasoning and proving. Deductive reasoning is unique in that it is the process of inferring conclusions from the known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information, and there is no need to validate them by experiments (Ayalon & Even, 2008). Although there are also other accepted forms of mathematical proving, a deductive proof is still considered as the preferred tool in the mathematics community for verifying mathematical statements and showing their universality (Hanna, 1990; Mariotti, 2006; Yackel & Hanna, 2003). On the other hand, inductive reasoning is also a very prominent manner of scientific thinking, providing for mathematically valid truths on the basis of concrete cases. Pólya (1967) indicates that inductive reasoning is a method of discovering
properties from phenomena and of finding regularities in a logical way, whereby it is crucial to distinguish between inductive reasoning and mathematical induction. Mathematical induction (MI) is a formal method of proof based more on deductive than on inductive reasoning. Some processes of inductive reasoning are completed with MI, but this is not always the case (Cañadas & Castro, 2007). Stylianides (2008, 2008a) uses the term reasoning-and-proving (RP) to describe the overarching activity that encompasses the following major activities that are frequently involved in the process of making sense of and in establishing mathematical knowledge: identifying patterns, making conjectures, providing non-proof arguments, and providing proofs. Given that RP is central to doing mathematics, many researchers and curriculum frameworks in different countries, especially in the United States, noted that a viable school mathematics curriculum should provide for the activities that comprise RP central to all students’ mathematical experiences, across all grade levels and content areas (Ball & Bass, 2003; Schoenfeld, 1994; Yackel & Hanna, 2003).

INDUCTIVE REASONING

As our research shall be dedicated to inductive reasoning, this will be specified from the perspectives of various theories and practices. Glaser and Pellegrino (1982, p. 200) identified inductive reasoning, as follows: »All inductive reasoning tasks have the same basic form or generic property requiring that the individual induces a rule governing a set of elements.« Inductive reasoning tasks can be solved either by applying the analytic strategy or the heuristics strategy (Klauer & Phye, 2008). The former enables one to solve every kind of an inductive reasoning problem. Its basic core would be the comparison procedure. The objects (or, in case of correlations, the pairs, triples, etc., of objects) would be checked systematically, predicate by predicate (attribute by attribute or relation by relation), in order to establish commonalities and/or diversities. However, the solution seekers generally tend to resort to the heuristics strategy, at which a participant starts with a more global task inspection and constructs a hypothesis, which can then be tested, so that the solution might be found more quickly, depending of the quality of the hypothesis. We believe that problem solving in mathematics is based on both strategies, with pupils, who learn mathematics, as well with scientists, who can reach new cognitions by applying either the analytic strategy or the heuristics one.

There are various theories as to the detailed identification of the stages of inductive reasoning. Pólya (1967) indicates four steps of the inductive reasoning process: observation of particular cases, conjecture formulation, based on previous particular cases, generalization and conjecture verification with new particular cases. Reid (2002) describes the following stages: observation of a pattern, the conjecturing (with doubt) that this pattern applies generally, the testing of the conjecture, and the generalization of the conjecture. Cañadas and
Castro (2007) consider seven stages of the inductive reasoning process: observation of particular cases, organization of particular cases, search and prediction of patterns, conjecture formulation, conjecture validation, conjecture generalization, general conjectures justification. There are some commonalities among the mentioned classifications: Reid (2002) believes the process to complete with generalization, Polya adds the stage of »conjecture verification«, as well as Cañadas and Castro (2007), who name the final stage “general conjectures justification”. In their opinions general conjecture is not enough to justify the generalization. It is necessary to give reasons that explain the conjecture with the intent to convince another person that the generalization is justified. Cañadas and Castro (2007) divided the Polya's stage of conjecture formulation into two stages: search and prediction of patterns and conjecture formulation.

The above stages can be thought of as levels from particular cases to the general case beyond the inductive reasoning process. Not all these levels are necessarily present; there are a lot of factors involved in their reaching. Pólya also states that induction, analogy and generalization are very close to each other. By observing and investigating special cases we notice similarities, regularities based on analogy and finally we state that the observed, noticed regularity yields in general case too.

**EMPIRICAL PART**

**Problem Definition and Methodology**

In the empirical part of the study conducted with primary teacher students and mathematics teacher students the aim was to explore their competences in inductive reasoning. In the early school years inductive reasoning is often used as a strategy to teach the basic mathematical concepts, as well as to solve problem situations. In the very research the focus was on the use of inductive reasoning at solving a mathematical problem. We believe that in mathematics only teachers who have competences in problem solving can create and deal with the situations in the classroom which contribute to the development of those competences in children.

The empirical study was based on the descriptive, non-experimental method of pedagogical research.

**Research Questions**

The aim of the study was to answer the following research questions:

1. Do the students possess adequate knowledge to solve the problem by applying the inductive reasoning strategy?
2. How much do the students delve into problem solving, i.e. which step in the process of inductive reasoning do they manage to take?
3. Which strategies are used by the students at their search for problem generalizations?
4. Is there any difference in the achieved problem solving depth and in applied strategies between primary teacher students and mathematics teacher students?
5. Are all the applied strategies equally effective for making generalizations?

Sample Description
The study was conducted at the Faculty of Education, University of Ljubljana, Slovenia in May 2010. It encompassed 89 third-year students of the Primary Teacher Education and 72 first-year students of Mathematics Teacher Education programme.

Data Processing Procedure
The students were posed a mathematical problem which was provided for the use of inductive reasoning in order to reach a solution and make generalizations. The problem was, as follows:

On the picture below the shaping of the spiral in the square of 4x4 is presented. Explore the problem of the spiral length in squares of different dimensions.

The students were solving the problem individually, they were simultaneously noting down their deliberations and findings, they were also aided with a blank square paper sheet of, so they could delve into the problem by drawing new spirals.

The data gathered from solving the mathematical problem were statistically processed by employing descriptive statistical methods. The students' solutions were analysed from two different perspectives: from the perspective of the problem solving depth and from the perspective of the applied strategies. As some students tested various problem solving strategies, thus contributing more than one solution to the result analysis, the decision was made to use the number of the received solutions and not the number of the participating students as the basis for the analysis of the problem solving depth and of the strategies of solving. We received 95 solutions from primary teacher students and 76 solutions from mathematics teacher students. Six primary teacher students and
Comparison of competences in inductive reasoning

four mathematics teacher students contributed two different approaches to the problem solving task.

**Results and Interpretation**

In continuation the results are shown, which are analysed as to various observation aspects.

a) *The problem solving depth*

The received solutions were classified into many levels, which were graded as to the achieved problem solving depth:

Level 1: the record contains only the pictures of the spirals,

Level 2: the record contains the drawn spirals and the corresponding calculations of the lengths of the spirals,

Level 3: the record contains structured records of the lengths of the spirals, but only for those cases, that are graphically presented,

Level 4: the record contains structured records of the lengths of the spirals and the prediction of the result for the case, which is not graphically presented,

Level 5: the record contains also the prediction for the general case.

As obvious the transformation of the problem from the geometric to the arithmetic one, and consequently operating with numbers and not only with pictures of the spirals is witnessed not until one has reached the level 2. Taking into account the stages in inductive reasoning (Polya, 1967, Reid, 2002, Canadas and Castro, 2007) we can also state that all the students at the levels from 1 to 5 reached the stage »observation of particular cases«, yet they were not equally successful in the process of searching and predicting of patterns. Mere drawings of spirals and calculations of their lengths (the levels 1 and 2) did not provide for a deeper insight into the nature of the problem and for making a generalization for the spiral of any dimension. The level 3 may be considered a transitional stage. These students already knew that mere calculations would not suffice, so they tried to structure them, i.e. they analysed the calculated numbers, and tried to define a certain pattern and a rule, respectively. However, they considered this to be enough and did not try to make a rule for the “n”-number of times-steps. In these cases students were deliberating on a possible pattern just for the cases they were observing. In comparison with them the level 4 students were already thinking about a possible pattern for a non-observing case, but they were still not thinking about applying their pattern to all cases. According to Reid (2002) the students at the level 4 reached the stage of conjecture (with doubt). They were convinced about the right of their conjecture for those specific cases, but not for other ones (see also Canadas and Castro, 2007). Only those students who achieved the level 5 can be considered to have reached the stage called »generalization of the conjecture« according to Reid.
In the opinions of Canadas and Castro (2007) generalization is by no means the final stage in the inductive reasoning process. The final stage - general conjectures justification – includes a formal proof that guarantees the veracity of the conjecture, namely. Similar to the research conducted by Canadas and Castro (2007), also in our research none of the students recognised the necessity to justify the results. They interpreted the results as an evident consequence of particular cases, with no need of any additional justification to be convinced of its truth.

Table 1 shows the distribution of responses regarding the achieved problem solving depth. »Other« group comprises the responses of students who were eliminated from further analysis of the problem solving procedures due to their non-understanding of the instructions.

<table>
<thead>
<tr>
<th>Depth</th>
<th>Primary teacher students</th>
<th>Mathematics teacher students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of responses</td>
<td>Responses in percentage</td>
</tr>
<tr>
<td>Level 1</td>
<td>6</td>
<td>6%</td>
</tr>
<tr>
<td>Level 2</td>
<td>19</td>
<td>20%</td>
</tr>
<tr>
<td>Level 3</td>
<td>24</td>
<td>25%</td>
</tr>
<tr>
<td>Level 4</td>
<td>11</td>
<td>12%</td>
</tr>
<tr>
<td>Level 5</td>
<td>32</td>
<td>34%</td>
</tr>
<tr>
<td>Other</td>
<td>3</td>
<td>3%</td>
</tr>
<tr>
<td>Total</td>
<td>95</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 1: Distribution of the responses regarding the achieved problem solving depth.

A closer comparison of primary teacher students’ and mathematics teacher students’ achievements shows some differences regarding the generalisation of the problem solving situations:

- there were more primary teacher students in comparison to mathematics teacher students who did not notice any structure among the collected data, which prevented them from further exploration (level 2);

- there were more mathematics teacher students (almost two thirds) in comparison to primary teacher students (one third) who achieved the highest level of generalisation (level 5). According to the presented results it could be concluded that the mathematics teacher students have better abilities to see the relations among the numbers, and have more knowledge for solving problems with inductive reasoning.
In addition, it is interesting to compare the percentages of the students who achieved the levels 3 and 4 (37 % of primary teacher students and 20 % of mathematics teacher students): they did notice the structure of the number pattern, but they were not able to develop the general form even it was explicitly noticeable. Most likely either they did not know how to write their findings in a general form or they did not feel the need to upgrade their concrete findings with a general form. Similar conclusion was made also by Cooper and Sakane (1986) who investigated 8th-grade students’ methods of generalising quadratic problems where most of the students could not explicitly recognise that particular cases should be examined for the general rule; some of them claimed that a pattern of numbers was sufficient rule in and of itself. Nevertheless, we think that the percentage of the primary teacher students who reached level 3 or 4 is quite high, and may reflect the orientation of primary teacher education focusing on dealing with concrete situations.

b) Problem solving strategies.

The analysis of the modes of reasoning that the students applied at their search for generalizations revealed that it was possible to perceive the posed problem from various perspectives. Various problem perception modes are addressed as various solving strategies in continuation, out of which the ones that were encountered among the students’ solutions are presented in Table 2.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Strategy description</th>
<th>Generalization record</th>
</tr>
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<tbody>
<tr>
<td>1 – »squares« strategy</td>
<td>It is observed that the values of the lengths are obtained by squaring the lengths of the consecutive square (e.g. 15 = 16 – 1)</td>
<td>((n+1)^2 - 1)</td>
</tr>
<tr>
<td>2 – »product« strategy</td>
<td>It is observed that the length of the spiral is equal to the product of two numbers that differ for 2 (e.g. 15 = 5x3)</td>
<td>(n(n+2))</td>
</tr>
<tr>
<td>3 – »binomial« strategy</td>
<td>It is observed that the length of the spiral is calculated by adding the double length to the square of the square length (e.g. 15 = 3x3 + 2x3)</td>
<td>(n^2 + 2n)</td>
</tr>
<tr>
<td>4 – »difference« strategy</td>
<td>When observing the differences among the lengths of the spirals, it is obvious that the result is the</td>
<td>The difference between the spiral in the square with nxn dimensions and the consecutive spiral is (2n + 1) or in a recursive</td>
</tr>
</tbody>
</table>
sequence of odd numbers (e.g. from 1x1 square onwards the lengths of the spirals increase by 5, 7, 9, 11, 13, 15….)

manner:
\[ d_{nxn} = d_{(n-1) x(n-1)} + (d_{(n-1) x(n-1)} - d_{(n-2) x(n-2)}) + 2), \]
whereby the denotation \( d_{nxn} \) stands for the length of the spiral in the square with nxn dimensions.

5 – »sum« strategy
It is observed that the length of the spiral can be presented as the sum of individual even sections of the spiral (e.g. 15 = 1 + 1 + 2 + 2 + 3 + 3 + 3.

3n + 2(n-1) – 2(n-2) ... + 2x2 + 2x1

6 - »quadrilateral« strategy
It is observed that the length of the spiral equals four times the length of the square enlarged by the product of two numbers that differ for 2 (e.g. 15 = 4x3 + 1x3)

4n + n(n-2)

7 – »transformation strategy«
It is observed that in cases when the dimension of the square is an even number, spirals can be transformed in squares, the perimeters of which can be calculated.

4n +4(n-2) + 4(n-4) ...+ 4x2; n=2k, k ∈ N

Table 2: Description of the applied problem solving strategies

In continuation the students' selection of the strategies is presented. The strategy was evaluated only with the responses, achieving the depth of the levels 3, 4. or 5., i.e. of those students, who noted the length of the spiral in a structured record, as it was possible to define the applied strategy and the mode of reasoning, respectively, only with this record.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Primary teacher students</th>
<th>Mathematics teacher students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of responses</td>
<td>Responses in percentage</td>
<td>Number of responses</td>
</tr>
<tr>
<td>1 – squares</td>
<td>2</td>
<td>2%</td>
</tr>
</tbody>
</table>
Table 3: Distribution of the responses as regards the applied problem solving strategy

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Primary Students</th>
<th>Mathematics Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 – product</td>
<td>8 (8%)</td>
<td>3 (4%)</td>
</tr>
<tr>
<td>3 – binomial</td>
<td>12 (14%)</td>
<td>4 (5%)</td>
</tr>
<tr>
<td>4 – difference</td>
<td>28 (29%)</td>
<td>16 (21%)</td>
</tr>
<tr>
<td>5 – sum</td>
<td>16 (17%)</td>
<td>38 (50%)</td>
</tr>
<tr>
<td>6 – mixed</td>
<td>0 (0%)</td>
<td>1 (1%)</td>
</tr>
<tr>
<td>7 – transformation</td>
<td>1 (1%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Other</td>
<td>28 (29%)</td>
<td>14 (19%)</td>
</tr>
<tr>
<td>Total</td>
<td>95 (100%)</td>
<td>76 (100%)</td>
</tr>
</tbody>
</table>

Let us have a closer look of the results presented in Table 3.

- Among the primary teacher students the strategy where the students focused on the difference between the lengths of the neighbouring spirals (29%) prevails whereas among the mathematics teacher students this was the sum strategy where students focused on adding the lengths of the individual even length sections of the spiral (50%).

- Among the primary teacher students the distribution of the used strategies is wider then among mathematics teacher students (or in other words: the distribution of the used strategies is more steady for primary teacher students in comparison to the mathematics teacher students). We can see that the »product« and »binomial« strategies are more often used among primary teacher students. Two primary teacher students also noticed that there was a correlation between the lengths of the spirals and the squares of the natural numbers which was not noticed among did the mathematics teacher students.

- In the »Other« column (Table 3) the responses were placed at which it was not possible to consider the selected strategy (all of the students who did not reach even the level 3).

The analysis of the problem solving strategies helps us to make conclusions about the effectiveness of a particular strategy for creating generalisation. It is very important to realise that all strategies are not equally effective for making generalisation and that the context of the problem might (not) support generalisation (Amit and Neria 2008). Therefore, further research question can be posed in analysing solving strategies, such as: were all the strategies equally effective when searching for generalizations?

The following table provides for the answer to this question, clarifying the relation between the selected strategy and the problem solving depth.
### Table 4: Problem solving depths in relation to the problem solving strategy

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Primary teacher students</th>
<th>Mathematics teacher students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Level 5</td>
<td>Total</td>
</tr>
<tr>
<td>1 – squares</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2 – product</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>3 binomial</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>4 – difference</td>
<td>4</td>
<td>28</td>
</tr>
<tr>
<td>5 – sum</td>
<td>11</td>
<td>16</td>
</tr>
<tr>
<td>6 – mixed</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7 – transformation</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The values in the last column for primary teacher students and in the last column for mathematics teacher students attest to the percentage of the responses pertaining to the selected strategy of those students who managed to reach the final level, i.e. the generalization.

According to the results one of the applied strategies was substantially less effective than the others for the both groups of students, i.e. the strategy 4 – ‘difference strategy’. Since it was most often used strategy among the primary education students (see table 3), a conclusion can be reached that the lower percentage of the achieved generalization among the primary teacher students compared to the mathematics teacher students was also due to the choice of the strategy. From this perspective some of the strategies (e.g. strategies 2, 3 and 5) were much more useful for creating general form than the other ones (strategy 4).

Let us have a more detailed examination of the strategy which was used by the most primary teacher students and gave the least correct generalisation – the ‘difference strategy’. The reason for choosing that strategy by a lot of students might be that searching for the difference between consecutive numbers is a very basic and well known strategy for making a generalisation. It is not difficult to obtain a generalisation if we get a constant difference between consecutive numbers at the first level of difference in a number pattern. On the other hand, the generalisation on the basis of the difference between consecutive numbers can be much more difficult if it demands the generalisation by function of higher order (not linear). In our case, the generalisation of the number pattern in the presented problem with spirals is expressed as quadratic function and this is in
in our opinion the main reason for a low ratio of the students who succeeded in creating generalisation on the basis of ‘difference strategy’ (see table 4).

In addition, it is also worth analysing the most used strategy among the mathematics teacher students, i.e. the ‘sum strategy’. This strategy was used by 50 % of them and proved to be very effective for forming generalisation. The closer look at those generalisations gave us 4 levels of quality difference among the achieved generalisations.

Level 1: Generalisation with an error. A student performs a generalisation in a recursive form as a sum of the even lengths of the spiral but does not determine the last article in a form (6 students)

Level 2: Generalisation in a recursive form as a sum of the even lengths of the spiral: \((3n + 2(n-1) − 2(n-2) … + 2x2 + 2x1)\) (24 students)

Level 3: Generalisation with the sum symbol: \(3n + 2\sum_{k=1}^{n-1} k\) (3 students)

Level 4: Simplifying the sum by transforming it into some of the records recognised in the strategies 1, 2 or 3, i.e.: \(3n + 2(n-1) − 2(n-2) … + 2x2 + 2x1 = 3n + 2((n-1) +(n-2)+…+2 +1) = 3n + 2n(n-1)/2 = n^2 + 2n\) (2 students)

It is worth emphasising that all primary teacher students who used the ‘sum’ strategy and created generalisation (11 students) could be placed in level 2, i.e. generalisation in a recursive form as a sum of the even lengths of the spiral.

What can we learn from these results? According to Steele and Johanning (2004) we could learn that the different quality levels of forming generalisation are the result of different schemas of the learners. They found out that the students whose schemas were partially formed could not consistently or clearly articulate the generalizations and had more recursive unclosed forms of symbolic generalizations (e.g. \(n+(n-1)-(n-1)+(n-2)\) and not \(4n-4\)). If we compare their results with ours it could be concluded that only a few students (5%) who have chosen the ‘sum’ strategy, achieved the level of well-connected schema.

SUMMARY

In the course of their studies at the Faculty of Education one of the important competences to be developed with primary teacher students and mathematics teacher students is to qualify them to solve mathematical problems. We are aware of the fact that this field of expertise is often neglected in our primary schools, mostly in favour of consolidating the learning contents by calculations and attending to classical word problems. We believe that students – future teachers are the ones, to whom we should start to bring about changes of this mindset, and introduce the role of the problem situations as an indispensable part of mathematics lessons in elementary schools. The presented research provided us with some important responses as to the qualification of students for
problem solving by inductive reasoning. It was established that the majority of
the students usually perceive the given situation as a problem, however, their
abilities to delve into the problem are rather different: based on the stages of
inductive reasoning according to Polya (1967), Reid (2002) and Castaneda and
Castro (2007) it can be inferred that the students’ responses were mainly
pertaining to the following three stages: observation of particular cases,
searching for pattern and prediction, as well as generalization. We find it
important to establish that the stage an individual student manages to reach is
largely influenced by his strategy selection. Some strategies in the process
solving proved to be more effective than the other ones, from the perspective of
making generalizations. Participating students approached the problem situation
in a creative manner, as they applied seven strategies of different quality, and
they were highly motivated to deal with such problem situations; both facts
seem to be extremely encouraging from the perspective of their later role as
teachers of mathematics to the youngest children.

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The paper presents an analysis of Greek primary school teachers’ problem solving methods, with a focus on the type of generalisations produced. Our results show that although most working groups have formulated some partial conclusions, they did not manage to reach a higher level of generalisation by some kind of ‘shift’ in their attention. Moreover, their works have demonstrated their reluctance in the use of mathematical notation in the form of variables and formulas.

INTRODUCTION

Generalisation is considered one of the most important processes involved in mathematics. Whether it is viewed as part of a higher level process, like abstraction (Dreyfus, 1991b) or as the core process involved in a particular mathematics field, like algebra (Mason, 1996), there seems to be an agreement on its significant role in advanced mathematical thinking. Moreover, significant curriculum documents, like NCTM’s Principles and Standards for School Mathematics (2000) state that:

Students should enter the middle grades with the view that mathematics involves examining patterns and noting regularities, making conjectures about possible generalizations, and evaluating the conjectures. In grades 6-8 students should sharpen and extend their reasoning skills by deepening their evaluations of their assertions and conjectures and using inductive and deductive reasoning to formulate mathematical arguments. (p. 262)

In accordance with the above, recent research has shown that even young children may engage in forms of generalisation (Lins & Kaput, 2004). Accepting that such processes can be introduced at an early age, it is vital to consider teachers’ education and how they should be prepared for initiating their students into algebraic reasoning. Firstly, one can make a distinction between secondary and primary school teachers, based on the premise that the former are expected to having been involved in advanced mathematical processes during their university studies. Indeed, Van Dooren, Verschaffel and Onghema (2003) have shown the different problem solving strategies followed by future primary and secondary school teachers in Flanders and the reluctance of some of the former to use algebraic methods. Our research stems from a similar need: in the context of Greek primary school teacher education, we aimed to analyse the
An analysis of pre-service teachers’ problem solving by generalisation

student-teachers’ problem solving methods, in order to examine the extent of their use of generalisation, which were expected as a solution to the problem posed. Particularly, our research questions were the following:

- How did the students interpret the task’s request for a (general) relation?
- What were the basic characteristics of the solution processes followed by the students?
- What form of representations did the students use to solve the problem and to present their answer?

THEORETICAL FRAMEWORK

According to Kaput (1999) algebraic thinking consists of: (a) the use of arithmetic as a domain for expressing and formalizing generalizations; (b) generalizing numerical patterns to describe functional relationships; (c) modelling as a domain for expressing and formalizing generalizations; and (d) generalizing about mathematical systems abstracted from computations and relations. The strong bonds between generalisation and mathematics (especially algebra) are quoted by numerous other researchers. Lee (1996) states that:

… it is possible to make a case for introducing algebra through functions, and through modeling, and through problem solving, quite as honestly as it is to make the case that generalizing activities are the only way to initiate students into the algebraic culture. (p. 102, our emphasis)

The various processes involved in generalisation have been identified by a number of researchers; Rivera and Becker (2008) in their literature review state that the initial stages in generalization involve: focusing on (or drawing attention to) a possible invariant property or relationship, ‘grasping’ a commonality or regularity and becoming aware of one’s own actions in relation to the phenomenon undergoing generalization. Mason (1996) offers an interpretive overview of these phases by seeing them as forming a spiralling helix, which contains:

- manipulation (whether of physical, mental, or symbolic objects) provides the basis for getting a sense of patterns, relationships, generalities, and so on;
- the struggle to bring these to articulation is an on-going one, and that as articulation develops, sense-of also changes;
- as you become articulate, your relationship with the ideas changes; you experience an actual shift in the way you see things, that is, a shift in the form and structure of your attention; what was previously abstract becomes increasingly, confidently manipulable. (pp. 81-82)

One of the basic tools that one has during generalising is visualization, i.e. a “process by which mental representations come into being” (Dreyfus, 1991b, p. 31); however, its use is not unproblematic for students who may be likely to
create visual images but are unlikely to use them for analytical reasoning (Dreyfus, 1991a). Generally, from the point of view of students, coming to think algebraically is not an easy process. The ‘shift of attention’ mentioned by Mason (1996) is the activity that differentiates the professional mathematician from the novice. Thus, the transition from arithmetic to algebra is a challenging aim for teachers in the last classes of primary school and the first of secondary school; and ‘early algebra’ is now a commonly used term (Lins & Kaput, 2004), signifying the assumption that the initiation to algebraic thinking may start in primary school:

... experiences in building and expressing mathematical generalizations – for us, the heart of algebra and algebraic thinking – should be a seamless process that begins at the start of formal schooling, not content for later grades for which elementary school children are “made ready”. (Blanton & Kaput, 2005, p. 35)

In order to clarify the teachers’ role in that initiation, and its consequent implications for teachers’ education, we could adopt a situated view of learning (Lave and Wenger, 1991), in which learning is seen as changing participation and formation of identities within relevant communities of practice. To put it simply, teachers should be initiated into the practices that they will initiate their students (Borko et al., 2005). Additionally, we are in line with Cobb (1994) who stresses that learning “should be viewed as both a process of active individual construction and a process of enculturation” (p. 13). In other words, we do not want to ignore the importance of engaging students in activities that are expected to promote the construction of meaningful knowledge. Bearing all these in mind we have designed a whole-semester teacher preparation programme, which forms the basis of the research presented in the paper.

**CONTEXT OF THE STUDY AND METHODOLOGY**

The teacher preparation programme in focus took place in the spring semester of 2011 at the Department of Primary Education of University of Ioannina in Greece. The participants of the course were 102 students in the third (out of four) year of their studies and both authors of the paper designed and realised the course. The course, entitled “Didactics of Mathematics I” is obligatory for all students and its intended aim is to provide the basic knowledge on contemporary theories for teaching and learning mathematics. Besides the lectures on the various approaches on mathematics education, the course included group-work activities, which aimed to improve our students’ basic mathematical competences (Niss, 2003), with a special focus on posing and solving mathematical problems and mathematical modelling (i.e. analysing and building models). Additionally, the students were initiated into a number of generalisation tasks, e.g. a variation of the ‘handshakes problem’ and the
matchsticks problem (e.g. Mason, 1996). The task presented here, taken from Dąbrowski (1993) aimed to further stimulate students’ mathematical investigations (Ponte, 2001) and eventually lead them to a generalisation; the type of the task called for visualisation, but in a simple form. An important characteristic of that task is that it can be implemented in differently aged students, allowing them to reach different levels of generalisation by observing and grouping the data. For example, primary school students are not necessarily required to reach a general formula for all the possible dimensions of the table. Concerning the way of working on the task Dąbrowski (1993) suggests group work as the optimum way, since it allows students to simultaneously consider different cases of the table’s dimensions.

![Figure 1. The billiard problem](image)

The problem is the following: In the billiard table shown we hit the ball from the bottom left hole at an angle of 45 degrees. The ball hits the table walls three times before it ends up in the top left hole. Thus, when the table’s dimensions are 3x2 the ball hits 3 times. What will happen if we hit the ball in the same way but in a table of different dimensions? Find the relationship between the table’s dimensions and the number of the ball hits.

The task was presented in the class and the students were given the opportunity to ask for any clarifications. They were then asked to form groups of two to four and work on the problem for around an hour. Their working sheets were collected and they comprise our main source of data.

The analysis of our data was done according to our research aim, i.e. to examine the generalisations reached by the students in their problem solving. Particularly, our data led us to focus on four aspects of the solutions, namely visualisation (C1), considered cases (C2), conclusions (C3) and formulas (C4). From these aspects, visualisation (C1) is the first part of the process of manipulation that we mentioned before, while the remaining three were informed by Mason’s (1996) view of generalisation as including variation, extension and pure generalisation. Particularly, the number and the type of considered cases (C2) were indicators

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4 It is important to note the big differences in our students’ level of mathematical knowledge which was mainly due to the Greek examination system which allows for students from different school specializations to enter the education university departments.
of the variety and the extent of students’ manipulations. The type of conclusions (C3) informed us on the extent of students’ articulations, while the type of formulas (C4) refers to the ‘shift of attention’ which is related to pure generalisation. These four aspects led us to the establishment of the following categories:

Category C1: Visualisation. It refers to the type of visualisations used and contains the following subcategories:

A. Orderly visualisation: contains the works in which the drawings were in some order, e.g. 4x3, 4x4, 4x5.

B. Non-orderly visualisation: contains the works in which the drawings were not clearly related to each other, e.g. 3x2, 4x5.

C. Multi cases visualisation: contains the works in which the considered situations were drawn on a single drawing.

Category C2: Considered cases. It refers to the number and the type of the cases considered and contains the following subcategories:

A. No (explicitly related) cases.

B. One case considered, the one where the billiard has the same dimensions (it forms a square).

C. The cases in which one dimension is constant, while the second dimension is changing.

D. The cases in which both dimensions are changing, having a fixed relation between them (e.g. the one is twice the other or they are two consecutive numbers).

E. The cases in which one dimension is an even (or an odd) number and the second dimension is changing.

F. The cases in which both dimensions are even (or odd) numbers.

G. The cases in which one dimension is an even and the other is an odd number.

H. The cases including any dimensions.

Category C3: Conclusions. It refers to the type of conclusions reached and contains the following subcategories:

A. Lack of conclusions.

B. Conclusion for a single case (I. correct, II. incorrect).

C. Conclusions for some cases (I. correct, II. incorrect).
D. Conclusions for all possible cases; works in which a number of conclusions appeared which were related to all particular cases, e.g. if \(m, n\) are even then... (I. correct, II. incorrect).

E. A general conclusion with some assumptions concerning the dimensions; works in which a general relation concerning the number of hits was provided, e.g. if \(m, n\) are the dimensions of the billiard table then the number of hits is described by the formula \(m+n-2\) (I. correct, II. incorrect).

**Category C4: Formulas.** It refers to the type of formulas reached and contains the following subcategories:

- **A. Lack of formulas.**
- **B. Formulas for only some of the considered cases (I. correct, II. incorrect).**
- **C. Different formulas for all the considered cases (I. correct, II. incorrect).**
- **D. A general formula for all cases (I. correct, II. incorrect).**

A more detailed description on the implementation of this analytical frame is given in the next section, where we present the results of this analysis.

**RESULTS**

Table 1 provides an overview of the way that our data were assigned to the categories described in the previous section. Additionally to the data shown below we examined the use of variables for the table’s dimensions. The columns marked in grey represent the sub-categories which were not finally related to any data.
Table 1. The initial data categorisation

Table 1 can be read in two ways, horizontally and vertically. By looking it horizontally one can follow a particular group’s work, i.e. observe the processes of manipulation and articulation (C1, C2 and C3) and whether the group has reached the level of pure generalisation (C4). For example, Group #8 has provided orderly drawings of the billiard table and some of them were done in the same drawing. That group has considered the following cases: a) the billiard has the same dimensions (it forms a square), b) one dimension is constant (equal to 2 and then to 3), while the second is changing (1, 2, 3, 4, ...), c) both dimensions are changing, having a fixed relation between them (the one is twice the other and then three times the other). For these cases – which do not account for all possible cases – the group has provided some correct conclusions, without the use of any formula. By looking at Table 1 vertically one can observe the strategies chosen by the students (C1 and C2) as well as the number of groups that formulated (correct) conclusions (C3) and formulas (C4). For example, by looking at the columns related to C4 we can notice that 26 groups did not formulate any formula (C4-A), while only one group formulated a formula for all cases, which was incorrect (C4-DII).

Sample analyses

An interesting result on the use of visualisations was that five groups did not make a single drawing (or they did not put it on their working sheet). In two of these cases it was apparent that the students did not make any drawing and this resulted in their solution process. A characteristic example is the work shown in Image 1, where the students suggest the use of proportions in order to calculate the number of hits of the ball. Particularly, we can see that the students are
initially calculating the area of the table in order to calculate the number of hits (“6 m\(^2\)→3 hits”). They then work on two cases (4x2 and 5x2) and after calculating the relevant number of hits by the use of proportions they write their first conclusion: “when the length is changing, the number of hits is the same with the length”. Then they examine the case of 3x3, which leads them to 4,5 (!) hits. The peculiarity of the non-natural number of hits does not prevent them from formulating their second conclusion that “the width is changing and defines the number of hits according to if it is even or odd number”.

The majority of student groups (19) have considered the cases in which one dimension is constant and the other is changing (Category C2-C). With the exception of one group, this led them to a conclusion at least for one case (Category C3-B). A characteristic example of a work belonging only to the C2-C category is shown in Image 2:
In the work above we see that the students have made their drawings in a single figure (C1-C) and they extended the table by one dimension each time. In their work we read: “If I increase the length of the large (side)” (and they consider the cases 3x2, 4x2, etc.) and then “If I increase the length of the small (side)” (and they consider the cases 3x3, 3x4, etc.). Their conclusions are written in the frame:

I observe that:

i) in an odd number for the length of the side the hits are equal to the length of the increasing side
ii) in an even number for the length of the side the only correlation is that the more the length of the side is being increased to the next even number, the more is incr... [the sentence is unfinished]

In the same category (C2) we can see that 10 groups have considered cases which were not explicitly related to each other (C2-A). Half of them did not manage to reach any conclusion; from the remaining five groups, four have reached incorrect conclusions (Works #11, 14, 20, 21). For example, in Work #14 the students have considered the cases 5x2, 4x2, 4x3 and 9x6 and their conclusion was that: “The more we increase the length of the billiard, the less the number of hits become”.

Another quite common solution found in 11 groups consisted of cases in which both dimensions were changing, but having a fixed relation between them (Category C2-D). The most frequent relation considered was that the dimensions are two consecutive numbers, e.g. 2x3, 3x4, 4x5, etc. In most cases that consideration led the students to correct conclusions, like the ones provided in Work #3:

- When one dimension is double from the other, it makes one (1) hit.
- When the dimensions are the same, it makes zero (0) hits.
- When the dimensions are even and their difference is 2 we have as many hits as the smaller dimension minus 1. For example, 4x2→1, 4x6→3, 6x8→5, 8x10→7
- When they are odd and their difference is 2 we have as many hits as the smaller dimension times 2 (x2). For example, 5x7→10 (2x5=10), 3x5→6 (2x3=6)
- When their difference is 1, i.e. when one is odd and the other even, the hits will be the double of the smaller dimension minus 1. For example, 3x2→3 ((2·2)-1=4-1=3), 4x3→5 ((3·2)-1=6-1=5), 7x8→13 (7·2)-1=14-1=13

It is noteworthy that only two groups provided more general cases for the table’s dimensions; particularly, one group (#34) considered all the possible cases for odd and even dimensions (Categories C2-F, C2-G) and another group (#29) considered the case for all possible numbers (Category C2-H). The work of the former of these groups is partially shown in Image 3. We have to note that apart from the worksheet with the printed version of the task, the particular group provided two more pages including more drawings and the relevant calculations which led them to their conclusions:
In the worksheet shown in Image 3 the conclusions are written in the top left and right sides and they quote that:

When $x$-even and $y$-odd then in each $y+1$ the ball will hit 2 times less

When $x$-odd and $y$-even in each $y+2$ then the hits are $y+1$ with the exception of the multiples of $y$ where the hits will be (assuming that $z$ is the multiple) –

When $x$-odd and $y$-even the hits are $y+1$

Although, as we already mentioned, the particular group has performed several calculations based on their drawings, their conclusions are all wrong; this could have been avoided by a process of verification.
The same was the situation concerning the use of variables and formulas (Category C4). Particularly, only eight groups used variables in their conclusions\(^5\), only five of them created some kind of formula and among them only one formula was correct, namely the formula \((n/2)-1\) provided by Group #22 for one dimension equal to 2 and the second dimension being an even number. An interesting case was Work #29 (Image 4 shows the main page in a total of three pages), which contains a ‘formula’ with two ‘variables’, for the length and the width: “For any number \(>\) width, \(\text{hits} = \text{length}+(\text{width} -1)\).” Here the group miscalculated the actual number of hits by one, since the ‘formula’ \(\text{hits} = \text{length}+\text{width}-2\) is true if the greatest common divisor of the length and the width equals to one.

\[\text{Image 4. Work #29}\]

In the above work we read from the top to the bottom:

where \(v\) the dimensions of the length

if \(v\) multiple of 2 then \(v/2\)

\(^5\) Note that the use of variables is not presented in Table 1.
if \( \nu \) odd \( \text{hits} = \nu +1 \)
if multiple of 3 \( \text{hits} = \nu/3 \)
any other number \( \text{hits} = \nu +2 \)
only for \( \nu =1 \rightarrow 3 \)
if multiple 4 \( \rightarrow \text{hits} = \nu/4 \)
other number \( \rightarrow \text{hits} = \nu +3 \)
except \( \nu =1 \rightarrow 4 \text{ hits} \)
\( \nu =2 \rightarrow 2 \text{ hits} \)

Inside the frame we read:

so for square \( \text{hits} = \text{length}/2 \)

For any number \( > \) width

\( \text{hits} = \text{length}+(\text{width}-1) \)

Finally, we have to note that no group reached the formula which represents the number of the ball hits for any table dimensions, which is:

\[ \left( \frac{n+m}{GCD(n,m)} \right) - 2, \]

where \( n, m \) stand for the table’s dimensions and \( GCD(n,m) \) is the Greatest Common Divisor of \( n, m \).

**CONCLUSIONS**

The study presented and the chosen task aimed to examine our pre-service teachers’ ability to generalise. Concerning our research questions, initially we may say that the students were much engaged in the task and all of them provided an answer to the question posed. During the process we had to explain to some of the students that they have to consider the number of hits for any dimensions. Finally, most groups provided the answer for the case of a square table; additionally, many groups provided a solution for one fixed and the second dimension of the table changing.

However, it seems that the expected shift in the form and structure of attention did not take place in most cases. This could be attributed to the students’ interpretation of the task; in other words, for some groups their ‘partial’ solutions were adequate, since they handed their worksheets quite early. In Mason’s (1996) words: “Generality is not a single notion, but rather is relative to an individual’s domain of confidence and facility. What is symbolic or abstract to one may be concrete to another” (p. 74). Thus, our students were not fully able to stress the important aspects and ignore the unimportant aspects of their data; this in turn may be attributed to the nature of the task’s data: in a first look the students were faced with a sequence of increasing and decreasing number of hits, not following an ‘obvious’ pattern.

Concerning the solution processes followed we may note that most of them were based on the following scheme: visualisation (drawings of related cases),
observation and articulation of regularities, and, finally, articulation of a conclusion. It is noteworthy that we have not seen in any paper a table for gathering the data, thus making it easier to study. What was also missing – or not provided in the worksheets – was any process of verification. The students seemed rather ‘easily’ (i.e. after examining very few cases) convinced on the validity of their statements and this eventually led some of them to wrong conclusions.

Apart from the drawings made as part of the initial phase of the solution, the students showed a clear preference on written descriptions of their considered cases, usually accompanied by mathematical expressions. In their conclusions, as we already noted, there was a clear lack of mathematical notation in the form of variables and formulas. We consider this an indication of our students’ mathematical background, which hindered them from the articulation of a ‘pure’ mathematical expression.

All the above call for a need for a more focused approach to generalisation in teachers’ education, preferably in the form of tasks that require not only a variety of manipulations but also some decision making by the students on the handling and interpreting data.

References


GENERALIZATIONS GEOMETRY IN ART ENVIRONMENT

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In this paper we describe activities leading to different types of generalizations of properties of polygons and symmetric patterns. Data shows that explorations and generalizations improve students’ learning and knowledge retention as well as their overall attitudes towards mathematics.

INTRODUCTION

Many recent studies demonstrate the power of explorations that integrate art and elementary mathematics. Loeb’s visual mathematics curriculum (Loeb, 1993), the “Escher World” project (Shaffer, 1997), and our earlier work (Grzegorczyk, Stylianou, 2005) show that mathematics learning was very effective in the context of arts-based lessons, and led to generalizations and abstraction on various levels. The National Council of Teachers of Mathematics (2000) also supports the introduction of extended projects, group work, and discussions to integrate mathematics across the curriculum.

In this study we presented three instructor-initiated explorations and discussion-based group activities leading to generalizations. Instead of starting with theoretical concepts, we introduced simple geometric examples to serve as a starting point to more complex, mathematical relationships. Students worked both individually and as a group and used art drawing and image-manipulation programs. The results of the study supported the main goal of this research, which was to show participants’ understanding of the generalized concepts and their strong knowledge retention. Additionally, we have observed increased positive attitudes towards mathematics.

METHODOLOGY

This study was conducted during the Mathematics and Fine Arts course in the arts studio environment with 21 students. The mathematical content of the course included the generation and analysis of artistic patterns, and the properties of polygons. During this study, students participated in three one-hour activities conducted during three 2-hour class sessions (the remaining time was used for testing, surveys and other issues not related to this study). Most of the participants had high-school level knowledge of mathematics, hence familiarity with algebraic formulas and geometric figures. Since the coursework involved the creation of artistic designs and patterns, the majority of participants were interested in fine arts. Table 1 below summarizes the initial characteristics of the participants. Note that participants majoring in Liberal Studies were prospective
elementary school teachers, while mathematics majors were prospective secondary school teachers.

<table>
<thead>
<tr>
<th>Interested in</th>
<th>Number of Major</th>
<th>Liked Art</th>
<th>Liked Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Art</td>
<td>7</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>Liberal Studies</td>
<td>7</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Mathematics</td>
<td>7</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>TOTAL</td>
<td>21</td>
<td>17</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 1: Initial description of participants.

This study was done as an introduction to tilings, tessellations and crystallographic groups. The following three-questions, 15-minute pre-test was given at the beginning of the first session to assess students’ initial knowledge.

P 1: What is the sum of the angles in a decagon?

P 2: How many diagonals does a decagon have? Justify your answer.

P 3: Draw a design that has a rotation by 120 degrees and at least one reflection.

Table 2 below summarizes the results of the pre-test for the group.

<table>
<thead>
<tr>
<th>Question</th>
<th>Correct answer</th>
<th>Correct Picture</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>P 1 (21)</td>
<td>4</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>P 2 (21)</td>
<td>3</td>
<td>14</td>
<td>0</td>
</tr>
<tr>
<td>P 3 (21)</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Pre-test results.

Note that only math majors gave correct answers on the pre-test. All sketches drawn to answer question P1 used a regular decagon subdivided into identical (central) triangles, which were later used for angle calculations. In P2 all students simply counted the diagonals.

The first two activities are based on the properties of polygons included in a typical high-school geometry course. However, they are presented as shortcut formulas for calculations, rather than used for building generalizing skills in students (Grzegorczyk, 2000). The first activity stresses recursive thinking and the second generalizes a counting algorithm. The third activity requires transfer of geometric properties (symmetries in this case) to non-mathematical objects (artistic images).

Discussion-Explorations-Generalization structure was used in all three activities. Instructor kept students focused on the following tasks.
1. Introduction of an initial problem.
2. Group discussion of the problem and possible generalizations.
3. Individual explorations of slightly generalized cases.
4. Group discussion of exploration results, methodologies and cases.
5. Verbalization of further generalization of the initial problem.
7. Group discussion of the proposed solutions and testing.
8. Verbalizing of the final generalized statement and further testing.

**First Activity – The sum of the interior angles of a polygon.**

1. Initial problem: *What is of the sum of angles in a triangle?* All students knew this sum is 180 degrees, but they could not justify. Instructor introduced the geometric proof represented graphically in Figure 2.

![Figure 2](image)

Figure 3. Angels in a triangle add up to a straight line.

2. Students discussed the proof and decided the argument would work for all triangles. Then the question was raised about quadrilaterals. Students agreed that the sum of the angles should be 360 degree based on their knowledge of rectangles. But they were not sure about other quadrilaterals.

3. First generalized problem: *What is of the sum of angles in a quadrilateral?* Students explored special cases: rhombus, parallelogram, and trapezoid. They generalized the triangle construction to quadrilaterals with two parallel sides.

4. Further discussion led to the idea of ‘cutting’ a quadrilateral into triangles and adding the angles, as shown in Figure 4. The case of non-convex quadrilaterals was raised and resolved. Hence, the solution to the first generalized question was established.
5. Further generalization: *What is the sum of angles in a pentagon?*

6. Students tried to apply the idea of suitable cutting of polygons into triangles. Discussion led to systematic division of each pentagon into three triangles meeting at one vertex (that worked well for convex cases), see Figure 5. Suitable diagonal cuts always gave three triangles regardless of the shape of the pentagon. Students established the answer as 540 degrees.

7. Discussion led to further generalizations: *What is the sum of angles in a hexagon, heptagon, and octagon?* Students figured out the answers to be 720, 900, 1080 degrees respectively. Instructor summarized their results as follows.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of sides</th>
<th>Sum of angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td>180</td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>360</td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>540</td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>720</td>
</tr>
<tr>
<td>Heptagon</td>
<td>7</td>
<td>900</td>
</tr>
<tr>
<td>Octagon</td>
<td>8</td>
<td>1080</td>
</tr>
</tbody>
</table>

Table 6. Angle sums for polygons.
8. Students noticed that the sum increases by 180, as there is one more triangle in the next step. They compared the number of sides to the number of triangles in each polygon (in each case getting 2 less). Their generalized statement: *The sum of the angles in an n-gon is 180(n-2).* They tested on various cases (the sum of angles in 102-gon is 18,000!).

9. All students thought that the formula does not require proof (because ‘construction shows it is true’). Instructor used *Mathematical Induction* to prove the statement. While all the students actively participated in the first 8 steps of this activity, only 9 (including all math majors) were interested in the proof.

Note that this activity requires *recursive thinking.* Students have discovered a universal truth (a theorem) about all polygons (even the ones that they did not consider in their explorations).

**Second Activity – The number of diagonals in a polygon.**

For simplicity students concentrated only on convex polygons in this activity.

1. Initial problem: *What is the number of diagonals in a (convex) hexagon?* All students could draw a hexagon and calculate diagonals as in Figure 7. Most of them colored them while counting.

![Figure 7. Diagonals in a hexagon.](image)

2. First generalization: *Calculate diagonals in pentagons, quadrilaterals and triangles.*

3. Discussion led to general questions of heptagons and octagons. All students calculated 14 and 20 diagonals respectively.

4. Discussion of various counting methods led to a generalized question: *Is there a connection between the number of sides and the number of diagonals?*

5. The group decided that since diagonals connect vertices, the number of sides is important. They collected their results in Table 8 below. *Is there a connection between the number of vertices and the number of diagonals?*
<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of sides</th>
<th>Number of vertices</th>
<th>Diagonals?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>Heptagon</td>
<td>7</td>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>Octagon</td>
<td>8</td>
<td>8</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 8. Vertices and diagonals in an $n$-gon.

7. Students looked for patterns in the table, a systematic way to express the relationship between the numbers of sides, vertices and diagonals. They noticed that sides already connect some vertices; hence only $n-3$ diagonals start at each vertex. They conjectured that there are $n(n-3)$ diagonals. Testing showed that they were overestimating. They noticed double counting, as each diagonal was counted for two vertices.

8. General statement was formulated as: $n$-gon has $n(n-3)/2$ diagonals. Students tested it on previous results. They calculated the number of diagonals in a nonagon, decagon, and some other polygons.

9. Students justified their formula as follows: Each $n$-gon has $n$ corners and there are $(n-3)$ diagonals starting at each corner. Since each diagonal starts at two corners, it gets counted twice while we count corner by corner. Therefore there are $n(n-3)/2$ diagonals.

Note that in this activity students had to generalize their counting method. The final statement was based on the invented systematic counting procedure.

**Activity 3 – Classifying small artistic designs using symmetries.**

This activity was conducted after students were familiar with reflections and rotations. They used software that generated images with various symmetries.

1. Initial Problem: Describe symmetries (reflections and rotations) of a square. Students sketched the picture representing the symmetries (see Figure 9).

![Figure 9. Symmetries of a square {m1, m2, m3, m4, r90, r180, r270, id}.](image)
2. Slightly generalized problem: *Describe all symmetries an equilateral triangle, a regular hexagon and a regular octagon.*

3. Students worked on lists of symmetries for each figure and searched for patterns. They decided on two generalizations below.

4. Generalization 1: *n*-sided regular polygon has exactly *n* different reflections. Student tested a regular octagon and other polygons to confirm their claim.

5. Generalization 2: *n*-sided regular polygon has rotations generated by $360/n$ degrees and confirmed that by checking on sketched figures.

6. Student worked individually with a pattern generating software to analyze symmetries of images. Figure 10 below shows an example of an image that was analyzed.

![Figure 10](image.png)

Figure 10. A pattern with 12 reflections and a rotation by 30 degrees.

7. Discussion led to grouping of the images with polygons that had similar symmetries.

8. Students decided that images could be classified by their symmetries. Generalization statement: *There are infinitely many types of (small) images depending on number of reflections D0, D1, D2, ...Dn, and the polygons represent each type depending on number of reflections.* (Note these symmetries form dihedral groups in abstract algebra).

9. Instructor and the students tried to justify the classification statement.

In this activity students had to *generalize the classification system to include non-mathematical objects.* They applied the language of mathematics to describe properties of artistic designs.

**DATA COLLECTION AND ANALYSIS OF RESULTS**

A week after each activity (at the beginning of the next session), students were given two post-test questions. Their answers were evaluated for correctness of the response, properness of the images used, and the quality of their justification. The following tables display the questions as well as the number of credits given to each group of students for correct answers. Note that questions
1, 2, 3 correspond to pre-test questions P1, P2 and P3. Paired T-test for pre- and post-test questions 1, 2 and 3 showed (statistically) significant improvement of students’ knowledge. Questions 1a, 2a, 3a were modified questions 1, 2, and 3. T-test comparison with corresponding pre-test questions shows significant improvement as well. Hence the activities used were an effective learning method.

**Question 1:** What is the sum of the angles in a regular twelve-sided polygon?

<table>
<thead>
<tr>
<th>Major</th>
<th>Correct answer</th>
<th>Correct Picture</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Art (7)</td>
<td>6</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>Lib. Studies (7)</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Mathematics(7)</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td><strong>TOTAL (21)</strong></td>
<td><strong>20</strong></td>
<td><strong>20</strong></td>
<td><strong>18</strong></td>
</tr>
</tbody>
</table>

All students used the formula from the first activity. One student miscalculated, one did not have a picture. Justifications explained derivation of the formula. Overall the group was very successful answering this question, ca 95% correct.

**Question 2:** Two identical regular pentagons were glued along one of the sides. What is the sum of the angles of this new polygon?

<table>
<thead>
<tr>
<th>Major</th>
<th>Correct answer</th>
<th>Correct Picture</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Art (7)</td>
<td>6</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>Lib. Studies (7)</td>
<td>6</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>Mathematics(7)</td>
<td>7</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td><strong>TOTAL (21)</strong></td>
<td><strong>19</strong></td>
<td><strong>21</strong></td>
<td><strong>18</strong></td>
</tr>
</tbody>
</table>

All of the students could picture the situation, but some were confused by the fact that the octagon was not convex. They subdivided the figure into triangles, and most of them calculated angles of the triangles and added them rather than using the formula. Justifications explained this addition process.

**Question 3:** How many diagonals does a decagon have? Justify your answer.

<table>
<thead>
<tr>
<th>Major</th>
<th>Correct answer</th>
<th>Correct Picture</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Art (7)</td>
<td>5</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Lib. Studies (7)</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Mathematics(7)</td>
<td>7</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td><strong>TOTAL (21)</strong></td>
<td><strong>19</strong></td>
<td><strong>17</strong></td>
<td><strong>14</strong></td>
</tr>
</tbody>
</table>

Four students did not provide pictures at all. All the math majors used the formula from Activity 2. Two Art and two Lib. Studies students counted the
Generalizations geometry in art environment

Four math majors introduced an algebraic formula for calculations. Over 80% of students were correct and close to 60% could justify their answers.

**Question 5:** Use letter P to create a design that has 4 reflections. Explain how to create this type of design, D4.

<table>
<thead>
<tr>
<th>Major</th>
<th>Correct Answer</th>
<th>Correct Picture</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Art (7)</td>
<td>7</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Lib. Studies (7)</td>
<td>5</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Mathematics (7)</td>
<td>7</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>TOTAL (21)</td>
<td>19</td>
<td>18</td>
<td>12</td>
</tr>
</tbody>
</table>

Two students sketched wrong designs, but one more did not show the lines of reflections on the correct picture. 10 pictures included a square in the background. In justification 8 students noted that ‘lines of reflections are like in a square’ and 4 said ‘mirror lines have to intersect at 45-degrees’.

**Question 6:** Use letter P to create a design that has a rotation by 60 degrees and at least one reflection.

<table>
<thead>
<tr>
<th>Major</th>
<th>Correct answer</th>
<th>Correct Picture</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Art (7)</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Lib. Studies (7)</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Mathematics (7)</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>TOTAL (21)</td>
<td>13</td>
<td>12</td>
<td>9</td>
</tr>
</tbody>
</table>
This question confused many students, as some of them tried to create a design with one reflection and the 60-degree rotation, which is impossible. Justifications included statements like ‘the design has to be like hexagon’, ‘the design has to have 6 reflections’. 11 students did not justify, while one art major with a correct design just said ‘since the angles are 60 degrees – it works’.

**Attitude evaluation**

At the end of the study, students re-evaluated their attitudes towards mathematics. Below is a summary of their responses. Note that the positive attitude towards mathematics increased from less than 50% (see Table 1) to over 70%. Almost all students had a positive experience with the software and over 85% liked the explorations. Interestingly, one math major was not happy with the activities.

<table>
<thead>
<tr>
<th>Do you like</th>
<th>Graphing software</th>
<th>Explorations</th>
<th>Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Art (7)</td>
<td>7</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Lib. Studies (7)</td>
<td>7</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Mathematics(7)</td>
<td>6</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>TOTAL (21)</td>
<td>20</td>
<td>18</td>
<td>15</td>
</tr>
</tbody>
</table>

**CONCLUSIONS**

The results of this study show that students of various interests and backgrounds can successfully be involved in mathematical explorations and generalizations. This particular group of students was able to think recursively, invent a counting method and transfer the classification criteria from simple geometric objects to an uncountable amount of designs. The numerical results show an improvement of students’ knowledge, more frequent use of formulas, and their ability to recover and verbalize the methods used to discover them. Additionally, almost the entire group liked the art-studio environment and graphics as a basis of learning. We observed an improvement of the general attitude towards mathematics among students with various interests. Students also commented that they enjoyed discussions, explorations and social ways of learning untypical in mathematics courses. Since the mathematical content of these activities is accessible to many younger students some modification or simplification of these explorations and generalizations activities may be successful in earlier grades.

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